

BKL CONJECTURE IN BIANCHI VIII AND IX WITH THE ULTRARELATIVISTIC FLUID

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ABSTRACT. We rigorously verify that in the spatially homogeneous spacetimes (Bianchi VIII and IX), the presence of matter does not affect the oscillatory behavior of the solutions to the Einstein field equations as first conjectured in [1]. This paper is an extension of [2] and uses the same formalism. We use the ultrarelativistic equation of state.

CONTENTS

Overview	2
1. Primary system of equations	4
1.1. Putting in the relativistic fluid	5
2. Reformulation of the conservation equation	8
3. Construction of transfer maps	9
4. Era-to-era and epoch-to-epoch maps	37
5. Abstract semi-global existence theorem	49
6. Main Theorems	49
References	60

OVERVIEW

We rigorously analyse the system of ordinary differential equations (ODE)

$$(1) \quad -\frac{d}{d\tau}\alpha_i - \beta_i^2 + \beta_j^2 + \beta_k^2 - 2\beta_j\beta_k + \frac{2}{3}\gamma^2 = 0$$

$$(2) \quad -\frac{d}{d\tau}\beta_i + \beta_i\alpha_i = 0$$

$$(3) \quad \frac{d}{d\tau}\gamma - \frac{1}{6}\gamma(\alpha_1 + \alpha_2 + \alpha_3) = 0$$

with $(i, j, k) \in C = \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$, that are subject to the quadratic constraint

$$(4) \quad \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 - \beta_1^2 - \beta_2^2 - \beta_3^2 + 2\beta_1\beta_2 + 2\beta_2\beta_3 + 2\beta_3\beta_1 = 4\gamma^2$$

where $\alpha = \alpha(\tau)$, $\beta = \beta(\tau)$ and $\gamma = \gamma(\tau)$, and the construction is carried out on $\tau \in [0, +\infty)^1$. The equations (1)-(4) are equivalent to the spatially homogeneous Einstein field equations with the ultrarelativistic fluid, see Section 1 and the beginning of Section 3.

In 1970, Belinski, Khalatnikov and Lifshitz (BKL) conjectured that there is a generic class of spacelike singularities, near which generic spacetimes behave like spatially homogeneous spacetimes² [1]. The latter exhibit an oscillatory behaviour described by the Gauss map, which is known to be chaotic. BKL's idea is that there is a generic class of solutions to the Einstein field equations that are described by decomposing the semi-infinite time axis into an infinite number of finite subintervals and by using the Kasner evolution on each subinterval³. These subintervals they called epochs. BKL conjectured that the presence of an ultra relativistic fluid does not affect this regime. The latter statement is rigorously verified in the present paper.

We reduce the analysis of (1)-(4) to the analysis of a Poincaré map on a five-dimensional Poincaré section. Modulo scaling symmetry of the solutions to (1)-(4), the set of parameters is (\mathbf{h}, w, q, z) . The parameters \mathbf{h} , w and q already appear in the vacuum case, see [2]. The parameter z describes the fluid. The important observation of [1], that the transition period between the Kasner epochs is small compared to the duration of the epochs, is encapsulated in the smallness of the parameter \mathbf{h} . Using the scale invariance of the solutions to (1)-(4), every orbit of the 4-dimensional discrete dynamical system is lifted to a *unique* orbit of the 5-dimensional discrete dynamical system through the map Λ , see Proposition 3.3.

¹We assume the singularity to be located at $\tau = +\infty$.

²For spatially homogeneous spacetimes the Einstein field equations reduce to a system of ODEs.

³Such behaviour is also called Mixmaster dynamics.

The oscillatory behaviour of the solutions to (1)-(4) can be understood in terms of the dynamics of $\beta_1, \beta_2, \beta_3$. Within each epoch, the logarithms of $|\beta_1|, |\beta_2|, |\beta_3|$ are approximately linear functions with slopes $\alpha_1, \alpha_2, \alpha_3$, with one component of α being positive and the other two negative at any point. At any time, at least two of the β 's are so small that they can be neglected to obtain the leading order approximation.

In Section 3, we show the existence of the transfer maps (i.e. maps from Poincaré section to Poincaré section), that map the state $\Phi(\tau_i)$ to an *earlier* state $\Phi(\tau_{i-1})$, where $\Phi = \alpha \oplus \beta \oplus \gamma \in \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}$ is a solution to (1)-(4). We also show that the transfer maps (\mathcal{P}_L, Π) are close to the approximate transfer maps $(\mathcal{P}_L, \mathcal{Q}_L)$ in Definition 3.16, and give explicit error bounds. See Proposition 3.3.

In Section 4, we analyse the dynamics of the approximate transfer maps and show that part of it is related to the dynamics of the Gauss map. The Gauss map is the left-shift on the continued fraction expansion of w . We use the domain of definition of the approximate transfer maps with full Lebesgue measure in w , where the continued fraction expansion grows at most polynomially. See Proposition 4.4.

In Section 6, we construct semi-global solutions by combining the results of Proposition 3.3 and the theorem in Section 5. See Theorems 6.2, 6.3.

Starting from Section 3, the notation and enumeration of the paper follows those in [2]. Every statement has an analogue for the vacuum case in [2].

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1. PRIMARY SYSTEM OF EQUATIONS

We start with arbitrary frame fields e_0, e_i, e_j, e_k , where (i, j, k) belong to the set of cyclic permutations of $(1, 2, 3)$. The goal is then to construct a spacetime, given that some $\alpha_i = \alpha_i(\tau)$ and $\beta_i = \beta_i(\tau)$ are *defined* through their commutators (no summation convention):

$$(5) \quad [e_j, e_k] =: e^\zeta \beta_i e_i$$

and similarly for α

$$(6) \quad [e_0, e_i] =: -\frac{1}{2} e^\zeta \alpha_i e_i,$$

where ζ is some (not yet explicitly defined) function. Ensuring that the Jacobi Identity holds brings

$$(7) \quad 0 = -\frac{d}{d\tau} \beta_i + \beta_i \alpha_i$$

which is exactly the original equation (1.1b) in [2].

Now, assuming that e_i are an orthonormal frame, we have the relation

$$g(\nabla_{e_a} e_b, e_c) = \frac{1}{2} (-g(e_a, [e_b, e_c]) - g(e_b, [e_a, e_c]) + g(e_c, [e_a, e_b]))$$

Further,

$$(8) \quad R_{abcd} = g([\nabla_{e_c}, \nabla_{e_d}]e_b - \nabla_{[e_c, e_d]}e_b, e_a)$$

Therefore, for the *vacuum* equations we get

$$-\frac{d}{d\tau} \alpha_i - \beta_i^2 + \beta_j^2 + \beta_k^2 - 2\beta_j \beta_k = 0$$

and the constraint

$$\alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_3 \alpha_1 - \beta_1^2 - \beta_2^2 - \beta_3^2 + 2\beta_1 \beta_2 + 2\beta_2 \beta_3 + 2\beta_3 \beta_1 = 0.$$

These are the exact equations that appear in [2].

1.1. Putting in the relativistic fluid. We have $R_{ab} = T_{ab} - \frac{1}{2}Tg_{ab}$, where T_{ab} is the energy-momentum tensor given by

$$(9) \quad T_{ab} = (p + \epsilon)u_a u_b + pg_{ab}$$

with pressure $p = p(\epsilon)$ and total energy density ϵ .

The equation (9) is the energy-momentum tensor of a perfect fluid. We consider the special case of the fluid at rest with a 4-velocity $u^a = (1, 0, 0, 0) = e_0$. Raising an index, we get $T^\mu_\nu = \text{diag}(-\epsilon, p, p, p)$. Therefore, for T (the trace of the energy-momentum tensor) we have

$$(10) \quad g^{ab}T_{ab} = T = T^0_0 + T^1_1 + T^2_2 + T^3_3 = 3p - \epsilon$$

Now, using the obtained results for the Ricci tensor components, we get:

$$(11) \quad R_{00} = T_{00} - \frac{1}{2}Tg_{00} = \frac{1}{2}(3p + \epsilon)$$

and (summation over i is not implied)

$$(12) \quad R_{ii} = T_{ii} - \frac{1}{2}Tg_{ii} = -\frac{1}{2}(p - \epsilon)$$

Also, $R_{0i} = 0$ and $R_{ij} = 0$ for $i \neq j$. Using the change of variables $P := e^{-2\zeta}p$ and $E := e^{-2\zeta}\epsilon$, the equation (12), the new version of the equation (1.1a) in [2] is

$$(13) \quad -\frac{d}{d\tau}\alpha_i - \beta_i^2 + \beta_j^2 + \beta_k^2 - 2\beta_j\beta_k = P - E$$

And the equation (1.1c) in [2], i.e. the constraint equation, is now

$$(14) \quad -\frac{d}{d\tau}(\alpha_1 + \alpha_2 + \alpha_3) + \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 = 3P + E$$

From now on we assume that $p = \frac{1}{3}\epsilon$, i.e. ultra relativistic equation of state.⁴

⁴The ultrarelativistic equation of state $p = \frac{1}{3}\epsilon$ follows directly from the general equation for gas pressure $p = \frac{1}{3} \int_0^\infty vkn(k)dk$, where $n(k)dk$ is the number density of particles in the momentum interval k to $k + dk$. Using $E = kc$ for ultrarelativistic particles, we get the equation.

Now, we also have the conservation equation for the energy-momentum tensor, i.e.

$$(15) \quad \nabla_\alpha T^{\alpha\beta} = 0$$

Substituting the definition for $T^{\alpha\beta}$ we get

$$(16) \quad \nabla_\alpha((p + \epsilon)u^\alpha u^\beta + pg^{\alpha\beta}) = (p + \epsilon)u^\alpha \nabla_\alpha u^\beta + g^{\alpha\beta} \nabla_\alpha p + u^\beta \nabla_\alpha[(p + \epsilon)u^\alpha] = 0$$

Using the fact that $u_\beta u^\beta = -1$ and contracting the equation above with u_β , we get

$$(17) \quad u^\alpha \partial_\alpha \epsilon + (p + \epsilon) \nabla_\alpha u^\alpha = 0$$

Using the definitions of P and E and the homogeneity assumption (i.e. only makes sense to take time derivatives) we obtain for the conservation equation:

$$(18) \quad e^\zeta \frac{d}{d\tau}(e^{2\zeta} E) + e^{2\zeta}(P + E) \frac{1}{2} e^\zeta (\alpha_i + \alpha_j + \alpha_k) = 0,$$

where $\nabla_{e_i} e_0 = \frac{1}{2} e^\zeta \alpha_i e_i$ was used to compute $\nabla_\alpha u^\alpha = \text{tr}(x \mapsto \nabla_x u) = \frac{1}{2} e^\zeta (\alpha_i + \alpha_j + \alpha_k)$. Now, using the definition of ζ , we have $\frac{d}{d\tau} \zeta = -\frac{1}{2} (\alpha_i + \alpha_j + \alpha_k)$. Substituting into (18), we get

$$(19) \quad e^\zeta (-e^{2\zeta} (\alpha_i + \alpha_j + \alpha_k) E + e^{2\zeta} \frac{d}{d\tau} E) + \frac{1}{2} e^{3\zeta} (P + E) (\alpha_i + \alpha_j + \alpha_k) = 0$$

which simplifies to

$$(20) \quad \frac{d}{d\tau} E + \frac{1}{2} (\alpha_1 + \alpha_2 + \alpha_3) (P - E) = 0.$$

Therefore, the new field equations, i.e. the new (1.1a) and (1.1b) are:

$$\begin{aligned} -\frac{d}{d\tau} \alpha_i - \beta_i^2 + \beta_j^2 + \beta_k^2 - 2\beta_j \beta_k &= P - E \\ -\frac{d}{d\tau} \beta_i + \beta_i \alpha_i &= 0. \end{aligned}$$

The new constraint (i.e. new (1.1c)) and conservation equation:

$$\begin{aligned}\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 - \beta_1^2 - \beta_2^2 - \beta_3^2 + 2\beta_1\beta_2 + 2\beta_2\beta_3 + 2\beta_3\beta_1 &= 4E \\ \frac{d}{d\tau}E + \frac{1}{2}(P - E)(\alpha_i + \alpha_j + \alpha_k) &= 0.\end{aligned}$$

By our ultrarelativistic assumption, the last equation simplifies to $\frac{d}{d\tau}E - \frac{1}{3}(\alpha_1 + \alpha_2 + \alpha_3)E = 0$.

These four equations are equivalent to the Einstein field equations for the relativistic fluid.

2. REFORMULATION OF THE CONSERVATION EQUATION

We have $\frac{d}{d\tau}\beta_i = \beta_i\alpha_i$. Therefore, $\frac{d}{d\tau}|\beta_i\beta_j\beta_k| = (\alpha_i + \alpha_j + \alpha_k)|\beta_i\beta_j\beta_k|$. Now, we can generalise it and write

$$(21) \quad \frac{d}{d\tau}(|\beta_i\beta_j\beta_k|^\chi) = \chi(\alpha_i + \alpha_j + \alpha_k)|\beta_i\beta_j\beta_k|^\chi$$

where χ is a constant. Next step is to show that (21) is equivalent to the conservation equation

$$(22) \quad \frac{d}{d\tau}E + \frac{1}{2}(P - E)(\alpha_i + \alpha_j + \alpha_k) = 0$$

We're working in Bianchi VIII and IX, therefore we can multiply both sides of (22) by $|\beta_i\beta_j\beta_k| \neq 0$. We get

$$\left(\frac{d}{d\tau}E\right)|\beta_i\beta_j\beta_k| + \frac{1}{2}(P - E)\underbrace{(\alpha_i + \alpha_j + \alpha_k)|\beta_i\beta_j\beta_k|}_{\frac{d}{d\tau}|\beta_i\beta_j\beta_k|} = 0$$

Using $P = \frac{E}{3}$ and (21), we find that $\chi = -\frac{1}{3}$, and we can write

$$\left(\frac{d}{d\tau}E\right)|\beta_i\beta_j\beta_k|^{-1/3} - \frac{1}{3}E(\alpha_i + \alpha_j + \alpha_k)|\beta_i\beta_j\beta_k|^{-1/3} = 0$$

which by the Leibnitz rule is equivalent to

$$(23) \quad \frac{d}{d\tau}(E|\beta_i\beta_j\beta_k|^{-1/3}) = 0.$$

3. CONSTRUCTION OF TRANSFER MAPS

Since E is a positive quantity, set $E =: \gamma^2$.

Introduce $n = (n_1, n_2, n_3, m)$. Then we have:

Definition 3.1. $\forall \Phi = \alpha \oplus \beta \oplus \gamma \in C^\infty((\tau_0, \tau_1), \mathbb{R}^6 \oplus \mathbb{R})$, $\forall \mathbf{h} > 0$, $\forall n \in \mathbb{R}^4$, associate a field

$$\mathfrak{a}[\Phi, \mathbf{h}, n] \oplus \mathfrak{b}[\Phi, \mathbf{h}, n] \oplus c[\Phi, \mathbf{h}, n] \oplus \mathfrak{d}[\Phi, \mathbf{h}, n] : (\tau_0, \tau_1) \rightarrow \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R} \oplus \mathbb{R}$$

by

$$(24) \quad \mathfrak{a}_i[\Phi, \mathbf{h}, n] = -\mathbf{h} \frac{d}{d\tau} \alpha_i - (n_i \beta_i)^2 + (n_j \beta_j - n_k \beta_k)^2 + \frac{2}{3} (m\gamma)^2$$

$$(25) \quad \mathfrak{b}_i[\Phi, \mathbf{h}, n] = -\mathbf{h} \frac{d}{d\tau} \beta_i + \beta_i \alpha_i$$

$$(26) \quad c[\Phi, \mathbf{h}, n] = -4(m\gamma)^2 + \sum_{(i,j,k)} (\alpha_j \alpha_k - (n_i \beta_i)^2 + 2n_j n_k \beta_j \beta_k)$$

$$(27) \quad \mathfrak{d}[\Phi, \mathbf{h}, n] = \mathbf{h} \frac{d}{d\tau} \gamma - \frac{1}{6} \gamma (\alpha_1 + \alpha_2 + \alpha_3)$$

where $(i, j, k) \in C$, set of cyclic permutations of $(1, 2, 3)$. Define

$$\mathfrak{a}_i[\Phi, \mathbf{h}, n_1, n_2] := \mathfrak{a}_i[\Phi, \mathbf{h}, n_1] - \mathfrak{a}_i[\Phi, \mathbf{h}, n_2]$$

Definition 3.2. Introduce the vectors $B_1 = (1, 0, 0, 0)$, $B_2 = (0, 1, 0, 0)$, $B_3 = (0, 0, 1, 0)$ and $Z = (1, 1, 1, 1)$ that will play the role of n .

Proposition 3.1. (GlobalSymmetries): Set $\chi(\tau) = p\tau + q$ with $p > 0$, then

$$(28) \quad (\mathfrak{a}, \mathfrak{b}, c, \mathfrak{d}) \left[A(\Phi \circ \chi), \frac{1}{p} A\mathbf{h}, n \right] = A^2 ((\mathfrak{a}, \mathfrak{b}, c, \mathfrak{d})[\Phi, \mathbf{h}, n] \circ \chi),$$

Remark 3.1. The equations $(\mathfrak{a}, \mathfrak{b}, c, \mathfrak{d})[\Phi, \mathbf{h}, Z] = 0$ are equivalent to (1)-(4) for any $\mathbf{h} > 0$.

Proposition 3.2. Recall Definition 3.1. For all $\Phi = \alpha \oplus \beta \oplus \gamma \in C^\infty((\tau_0, \tau_1), \mathbb{R}^6 \oplus \mathbb{R})$, all $\mathbf{h} > 0$, all $n \in \mathbb{R}^4$, we have

$$(29) \quad 0 = -\mathbf{h} \frac{d}{d\tau} c + \sum_{(i,j,k) \in C} (-\alpha_j \mathfrak{a}_k - \alpha_k \mathfrak{a}_j + 2(n_i)^2 \beta_i \mathfrak{b}_i - 2n_j n_k \beta_j \mathfrak{b}_k - 2n_j n_k \beta_k \mathfrak{b}_j) - 8m^2 \gamma \mathfrak{d}$$

Proof: Straightforward.

Definition 3.3. $\forall \mathbf{h} \in (0, \infty)$, $\forall \Phi = \alpha \oplus \beta \oplus \gamma$ with $\beta_1, \beta_2, \beta_3 \neq 0$ define

$$\begin{aligned} A_m[\Phi] &:= \sqrt{|\alpha_m|^2 + |\beta_m|^2} \\ \phi_m[\Phi] &:= -\operatorname{arcsinh} \frac{\alpha_m}{|\beta_m|} \\ \xi_m[\Phi, \mathbf{h}] &:= \mathbf{h} \log \left| \frac{1}{2} \beta_m \right| \\ \alpha_{m,n}[\Phi] &:= \alpha_m + \alpha_n, \quad \xi_{m,n} := \xi_m[\Phi, \mathbf{h}] + \xi_n[\Phi, \mathbf{h}] \\ \rho &:= \gamma |\beta_1 \beta_2 \beta_3|^{-1/6} \end{aligned}$$

with $m = 1, 2, 3$.

Remark 3.2. The same as Remark 3.2 in [2]

Lemma 3.1. The same as Lemma 3.1 in [2]

Definition 3.4. Set S_3 = the set of all permutations of $(1, 2, 3)$.

Definition 3.5. For all $\sigma_* \in \{-1, +1\}^3$ let $\mathcal{D}(\sigma_*)$ be the set of all $\Phi = \alpha \oplus \beta \oplus \gamma \in \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}$ with $(\operatorname{sgn} \beta_1, \operatorname{sgn} \beta_2, \operatorname{sgn} \beta_3) = \sigma_*$. For all $\tau_0, \tau_1 \in \mathbb{R}$ with $\tau_0 < \tau_1$ let $\mathcal{E}(\sigma_*; \tau_0, \tau_1)$ be the set of all continuous maps $\Phi : [\tau_0, \tau_1] \rightarrow \mathcal{D}(\sigma_*)$.

Definition 3.6. $\pi = (a, b, c) \in S_3$, $\mathbf{h} > 0$, $\sigma_* \in \{-1, +1\}^3$. Define $d_{\mathcal{D}(\sigma_*), (\pi, \mathbf{h})} : \mathcal{D}(\sigma_*) \times \mathcal{D}(\sigma_*) \rightarrow [0, \infty)$ by

$$\begin{aligned} d_{\mathcal{D}(\sigma_*), (\pi, \mathbf{h})}(\Phi, \Psi) = \max \{ & |A_a[\Phi] - A_a[\Psi]| \quad , \quad \left| \mathbf{h} \frac{\phi_a[\Phi]}{A_a[\Phi]} - \mathbf{h} \frac{\phi_a[\Psi]}{A_a[\Psi]} \right|, \\ & |\alpha_{b,a}[\Phi] - \alpha_{b,a}[\Psi]| \quad , \quad |\xi_{b,a}[\Phi, \mathbf{h}] - \xi_{b,a}[\Psi, \mathbf{h}]|, \\ & |\alpha_{c,a}[\Phi] - \alpha_{c,a}[\Psi]| \quad , \quad |\rho^2[\Phi] - \rho^2[\Psi]| \} \end{aligned}$$

and

$$\mathcal{A}_{\mathcal{D}(\sigma_*)}(\Phi, \Psi) = \max_{i=1,2,3} \{ |\alpha_i[\Phi] - \alpha_i[\Psi]|, |\xi_i[\Phi, \mathbf{h}] - \xi_i[\Psi, \mathbf{h}]|, |\rho^2[\Phi] - \rho^2[\Psi]| \}$$

Both $d(\Phi, \Psi)$ and $\mathcal{A}(\Phi, \Psi)$ satisfy the three defining properties of a metric ($d(\Phi, \Psi) \geq 0$, with equality iff $\Phi = \Psi$; $d(\Phi, \Psi) = d(\Psi, \Phi)$ and $d(\Phi, \Psi) \leq d(\Phi, \Psi') + d(\Psi', \Phi)$). Analogue for \mathcal{A}). Therefore, $(\mathcal{D}(\sigma_*), d_{\mathcal{D}(\sigma_*), (\pi, \mathbf{h})})$ and $(\mathcal{D}(\sigma_*), \mathcal{A}_{\mathcal{D}(\sigma_*), \mathbf{h}})$ are metric spaces.

Definition 3.7. Stays the same as Definition 3.7 in [2].

Lemma 3.2. Stays the same as in [2]. The addition of ρ^2 terms to both d and \mathcal{A} doesn't make a difference because $C, D \geq 1$ so both old inequalities a) and b) also hold for the case with the ultra relativistic fluid.

Definition 3.8. Let $\mathcal{X} = \mathcal{D}(\sigma_*)$ or $\mathcal{X} = \mathcal{E}(\sigma_*; \tau_0, \tau_1)$. For all $\delta \geq 0$ and $\Phi \in \mathcal{X}$ and $\pi \in S_3$ and $\mathbf{h} > 0$, set $B_{\mathcal{X}, (\pi, \mathbf{h})}[\delta, \Phi] = \{\Psi \in \mathcal{X} | d_{\mathcal{X}, (\pi, \mathbf{h})}(\Phi, \Psi) \leq \delta\}$.

Definition 3.9. (*Reference field*) Recall Definition 3.3. For all $\pi \in S_3$, $(\mathbf{h}, w, q, z) \in (0, \infty)^4$, the reference field $\Phi_0 = \Phi_0(\pi, (\mathbf{h}, w, q, z), \sigma_*)$ is given by

$$(30) \quad A_a[\Phi_0](\tau) = 1$$

$$(31) \quad \theta_a[\Phi_0, \mathbf{h}](\tau) = 0$$

$$(32) \quad \alpha_{b,a}[\Phi_0](\tau) = -(1+w)^{-1}$$

$$(33) \quad \alpha_{c,a}[\Phi_0](\tau) = -(1+w)$$

$$(34) \quad \xi_{b,a}[\Phi_0, \mathbf{h}](\tau) = -1 - \mathbf{h} \log 2 - (1+w)^{-1} \tau$$

$$(35) \quad \xi_{c,a}[\Phi_0, \mathbf{h}](\tau) = -(1+w)q - \mathbf{h} \log 2 - (1+w)\tau$$

$$(36) \quad \gamma[\Phi_0, h](\tau) = \gamma_0 \exp \left[-\frac{1}{\mathbf{h}} \frac{1}{6} \frac{2+w(2+w)}{1+w} \tau \right] \left(\text{Cosh} \left(\frac{1}{\mathbf{h}} \tau \right) \right)^{1/6}$$

Define $\sqrt{z} := \rho[\Phi_0]$

Lemma 3.3. Let Φ_0 be as in Definition 3.9. Then $(\mathbf{a}, \mathbf{b}, c, \mathbf{d})[\Phi_0, \mathbf{h}, B_a] = 0$.

Definition 3.10. $\forall \mathbf{f} = (\mathbf{h}, w, q, z) \in (0, \infty)^4$ set

$$\begin{aligned} \tau_-(f) &= - \left(1 - \frac{1}{2+w} \right) \min(1, q) < 0 \\ \tau_+(f) &= 1 + \frac{1}{w} > 0 \end{aligned}$$

Motivation for Definition 3.10. The bounce of β_a happens at $\tau = 0$, then the next bounce (towards the right) will be by β_b , i.e. $\beta_b = 1$ and $\xi_b = 0$ (per definition). This is exactly τ_+ . We have: $\xi_b = -1 - \mathbf{h} \log 2 - \frac{\tau}{1+w} - \xi_a \approx -1 - \mathbf{h} \log 2 - \frac{\tau}{1+w} - (-\tau)$. Therefore, setting $\xi_b = 0$ we get $\tau = \tau_+$. For τ_- we consider the bounce towards the left, i.e. for ξ_c , this refers to $\tau < 0$. By putting $\xi_c = 0$ and taking the approximation $\xi_a \approx \tau$, we get the τ_- . Note that in both cases we ignored the $\mathbf{h} \log 2$ terms as small.

Lemma 3.4. (*Technical Lemma 1*)

Let $\pi = (a, b, c) \in S_3$, $\mathbf{f} = (\mathbf{h}, w, q, z) \in (0, \infty)^4$, $\sigma_* \in \{-1, +1\}$. Fix $\delta > 0$, $\epsilon_- \in (0, -\tau_-)$, $\epsilon_+ \in (0, \tau_+)$ where $\tau_{\pm} = \tau_{\pm}(\mathbf{f})$. Set

$$\begin{aligned} \tau_{0-} &= \tau_- + \epsilon_- < 0 \\ \tau_{0+} &= \tau_+ - \epsilon_+ > 0 \\ \Phi_0 &= \Phi_0(\pi, \mathbf{f}, \sigma_*)|_{[\tau_{0-}, \tau_{0+}]} \\ \mathcal{E} &= \mathcal{E}(\sigma_*; \tau_{0-}, \tau_{0+}). \end{aligned}$$

Assume $\delta \leq 2^{-4} \min\{1, w, \epsilon_-, \frac{\epsilon_+}{\tau_+ \tau_{0+}}\}$ holds.

The estimates in [2] for β 's and their products still hold and are

$$\begin{aligned} |\beta_b \beta_a| &\leq 2 \exp \left(-\frac{1}{4\mathbf{h}} \right) \\ |\beta_c \beta_a| &\leq 2 \exp \left(-\frac{1}{2\mathbf{h}} \epsilon_- \right) \\ |\beta_b|^2 &\leq 2^4 \exp \left(-\frac{1}{\mathbf{h}} \min \{ \epsilon_-, \frac{\epsilon_+}{\tau_+} \} \right) \\ |\beta_c|^2 &\leq 2^4 \exp \left(-\frac{2}{\mathbf{h}} \epsilon_- \right) \end{aligned}$$

Therefore, we only need an estimate for $\gamma^2[\Phi]$.

We have

$$|\gamma^2[\Phi]| \leq \rho^2[\Phi] 2 \exp \left(-\frac{1}{4\mathbf{h}} \right) |^{1/3} 4 \exp \left(-\frac{1}{\mathbf{h}} \epsilon_- \right) |^{1/3} \leq 2z \exp \left(-\frac{1}{12\mathbf{h}} (1 + 4\epsilon_-) \right)$$

The new total estimate is then

$$\max \{ |\beta_b|^2, |\beta_c|^2, |\beta_b \beta_a|, |\beta_c \beta_a|, |\gamma|^2 \} \leq \max \{ 2^4, 2z \} \exp \left(-\frac{1}{12\mathbf{h}} \min \{ 1, \epsilon_-, \frac{\epsilon_+}{\tau_+}, 1 + 4\epsilon_- \} \right)$$

Since $\epsilon_- > 0$ per definition, we have $\epsilon_- < 1 + 4\epsilon_-$ always. Also, $1 + 4\epsilon_- \geq 3\epsilon_-$. Therefore,

$$\max \{ |\beta_b|^2, |\beta_c|^2, |\beta_b \beta_a|, |\beta_c \beta_a|, |\gamma|^2 \} \leq 2^4 \max \{ 1, z \} \exp \left(-\frac{1}{4\mathbf{h}} \min \{ 1, \epsilon_-, \frac{\epsilon_+}{\tau_+} \} \right).$$

Lemma 3.5. Recall Definitions 3.1 and 3.2 . $\forall (a, b, c) \in S_3$ we have

$$\begin{aligned} \mathbf{a}_a[\Phi, \mathbf{h}, Z, B_a] &= +\beta_b^2 + \beta_c^2 - 2\beta_b \beta_c + \frac{2}{3}\gamma^2 \\ \mathbf{a}_b[\Phi, \mathbf{h}, Z, B_a] &= -\beta_b^2 + \beta_c^2 - 2\beta_a \beta_c + \frac{2}{3}\gamma^2 \\ \mathbf{a}_c[\Phi, \mathbf{h}, Z, B_a] &= +\beta_b^2 - \beta_c^2 - 2\beta_a \beta_b + \frac{2}{3}\gamma^2 \end{aligned}$$

Remark 3.3. Lemma 3.5 gives the difference between $\mathbf{a}[\Phi, \mathbf{h}, Z] = 0$ (the actual field) and $\mathbf{a}[\Phi, \mathbf{h}, B_a] = 0$ (the reference field). The bounds for the terms are given in Technical Lemma 1, and tend exponentially to zero as $\mathbf{h} \rightarrow 0$. Note, however, that ρ was assumed to be some (finite) constant (but different for every field Φ).

Definition 3.11. $\Phi = \alpha \oplus \beta \oplus \gamma \in \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}$, $\beta_i \neq 0$, $\mathbf{h} > 0$. Define four real numbers by

$$\begin{aligned} \mathbf{I}_1[\Phi, \mathbf{h}, \pi] &= -\frac{1}{\mathbf{h}} \mathbf{a}_a[\Phi, \mathbf{h}, Z, B_a] \tanh \phi_a[\Phi] \\ \mathbf{I}_2[\Phi, \mathbf{h}, \pi] &= (A_a[\Phi])^{-2} \mathbf{a}_a[\Phi, \mathbf{h}, Z, B_a] (1 - \tanh \phi_a[\Phi]) \\ \mathbf{I}_{(3,p)}[\Phi, \mathbf{h}, \pi] &= \frac{1}{\mathbf{h}} \mathbf{a}_p[\Phi, \mathbf{h}, Z, B_a] + \frac{1}{\mathbf{h}} \mathbf{a}_a[\Phi, \mathbf{h}, Z, B_a] \end{aligned}$$

with $p \in \{\mathbf{b}, \mathbf{c}\}$.

Lemma 3.6. (Technical Lemma 2) Recall $\rho[\Phi_0] = \sqrt{z}$. In the context of Definition 3.11 the following estimates hold:

$$\begin{aligned} |\mathbf{I}_S[\Phi]| &\leq 2^{12} \max\{1, z\} \max\{1, \frac{1}{\mathbf{h}}, \frac{1}{\mathbf{h}}|\tau|\} \exp\left(-\frac{1}{4\mathbf{h}} \min\{1, \epsilon_-, \frac{\epsilon_+}{\tau_+}\}\right) \\ |\mathbf{I}_S[\Phi] - \mathbf{I}_S[\Psi]| &\leq 2^{18} \max\{1, z\} \left(\max\{1, \frac{1}{\mathbf{h}}, \frac{1}{\mathbf{h}}|\tau|\}\right)^2 \exp\left(-\frac{1}{4\mathbf{h}} \min\{1, \epsilon_-, \frac{\epsilon_+}{\tau_+}\}\right) d_{\mathcal{E}}(\Phi, \Psi) \end{aligned}$$

for all $S \in \{1, 2, (3, b), (3, c)\}$.

Proof. Set $M := \exp\left(-\frac{1}{4\mathbf{h}} \min\{1, \epsilon_-, \frac{\epsilon_+}{\tau_+}\}\right)$, $M_1 := \max\{1, \frac{1}{\mathbf{h}}, \frac{1}{\mathbf{h}}|\tau|\}$, and $M_2 := \max\{1, z\}$. For $x \geq a$ we have the inequality

$$\begin{aligned} |e^x - e^a| &= |e^a| \cdot |e^{x-a} - 1| \\ &= |e^a| \cdot \left| (x-a) + \frac{(x-a)^2}{2!} + \dots \right| \\ &= |e^a| \cdot |x-a| \cdot \left| 1 + \frac{x-a}{2!} + \frac{(x-a)^2}{3!} + \dots \right| \\ &\leq |e^a| \cdot |x-a| \cdot |e^{x-a}| \end{aligned}$$

It implies that $|e^x - e^y| \leq \max\{e^x, e^y\} |x - y|$, $\forall x, y$.

Recall $\log|\beta_i| = \frac{1}{\mathbf{h}}\xi_i + \log 2$.

Estimates. Lemmas 3.4 and 3.5 imply $|\mathbf{a}_i[\Phi, \mathbf{h}, Z, B_a]| \leq 2^7 M M_2$, for $i = 1, 2, 3$, and $\phi_a[\Phi] \leq 2^2 M_1$, and $(A_a[\Phi])^{-2} \leq 2^2$. Therefore, for the $|I_S|$ estimate we have $|I_1| \leq 2^7 M M_1 M_2$ and $|I_2| \leq 2^{12} M M_1 M_2$ and $|I_3| \leq 2^8 M M_1 M_2$.

Further, using Definition 3.3, Definition 3.6, we have:

$$|(\beta_a \beta_b \beta_c)[\Phi]|^{1/3} = 2 \exp\left(-\frac{1}{12\mathbf{h}}(1 + 4\epsilon_-)\right) \leq 2M$$

and

$$\begin{aligned} |\beta_p[\Phi]\beta_a[\Phi] - \beta_p[\Psi]\beta_a[\Psi]| &\leq \frac{1}{\mathbf{h}} \max\{|\beta_p[\Phi]\beta_a[\Phi]|, |\beta_p[\Psi]\beta_a[\Psi]|\} |\xi_{a,p}[\Phi] - \xi_{a,p}[\Psi]| \\ &\leq \frac{1}{\mathbf{h}} 2^4 M_2 M d_{\mathcal{E}}(\Phi, \Psi) \\ &\leq 2^4 M M_1 M_2 d_{\mathcal{E}}(\Phi, \Psi) \end{aligned}$$

and

$$\begin{aligned}
|\xi_a[\Phi] - \xi_a[\Psi]| &\leq \mathbf{h}|\log A_a[\Phi] - \log A_a[\Psi]| + \mathbf{h}|\log \cosh \phi_a[\Phi] - \log \cosh \phi_a[\Psi]| \\
&\leq \mathbf{h}|\log A_a[\Phi] - \log A_a[\Psi]| + \mathbf{h}|\phi_a[\Phi] - \phi_a[\Psi]| \\
&\leq \mathbf{h} \max\left\{\frac{1}{A_a[\Phi]}, \frac{1}{A_a[\Psi]}\right\} |A_a[\Phi] - A_a[\Psi]| + \mathbf{h}|\phi_a[\Phi] - \phi_a[\Psi]| \\
&\leq \frac{\mathbf{h}}{2} d_{\mathcal{E}}(\Phi, \Psi) + \mathbf{h} 2^2 M_1 d_{\mathcal{E}}(\Phi, \Psi) \\
&\leq 2^3 \mathbf{h} M_1 d_{\mathcal{E}}(\Phi, \Psi)
\end{aligned}$$

and

$$\begin{aligned}
|\beta_p[\Phi] - \beta_p[\Psi]| &\leq \frac{1}{\mathbf{h}} \max\{|\beta_p[\Phi]|, |\beta_p[\Psi]|\} |\xi_p[\Phi] - \xi_p[\Psi]| \\
&\leq \frac{1}{\mathbf{h}} 2^2 (M_2 M)^{1/2} (|\xi_{a,p}[\Phi] - \xi_{a,p}[\Psi]| + |\xi_a[\Phi] - \xi_a[\Psi]|) \\
&\leq \frac{1}{\mathbf{h}} 2^2 (M_2 M)^{1/2} (d_{\mathcal{E}}(\Phi, \Psi) + |\xi_a[\Phi] - \xi_a[\Psi]|) \\
&\leq 2^2 (M M_2)^{1/2} \left(\frac{1}{\mathbf{h}} + 2^3 M_1 \right) d_{\mathcal{E}}(\Phi, \Psi) \\
&\leq 2^6 (M M_2)^{1/2} M_1 d_{\mathcal{E}}(\Phi, \Psi)
\end{aligned}$$

and

$$\begin{aligned}
|\phi_a[\Phi] - \phi_a[\Psi]| &\leq \frac{1}{\mathbf{h}} |A_a[\Phi] - A_a[\Psi]| |\tau| + \frac{1}{\mathbf{h}} |A_a[\Phi] \theta_a[\Phi] - A_a[\Psi] \theta_a[\Psi]| \\
&\leq \frac{1}{\mathbf{h}} (1 + |\tau|) |A_a[\Phi] - A_a[\Psi]| + \frac{1}{\mathbf{h}} 2 |\theta_a[\Phi] - \theta_a[\Psi]| \\
&\leq \frac{1}{\mathbf{h}} (3 + |\tau|) d_{\mathcal{E}}(\Phi, \Psi) \\
&\leq 2^2 M_1 d_{\mathcal{E}}(\Phi, \Psi)
\end{aligned}$$

and

$$\begin{aligned}
|\rho^2[\Phi](\beta_a\beta_b\beta_c)^{1/3}[\Phi] - \rho^2[\Psi](\beta_a\beta_b\beta_c)^{1/3}[\Psi]| &\leq |\rho^2[\Phi](\beta_a\beta_b\beta_c)^{\frac{1}{3}}[\Phi] - \rho^2[\Psi](\beta_a\beta_b\beta_c)^{\frac{1}{3}}[\Phi]| \\
&+ |\rho^2[\Psi](\beta_a\beta_b\beta_c)^{\frac{1}{3}}[\Phi] - \rho^2[\Psi](\beta_a\beta_b\beta_c)^{\frac{1}{3}}[\Psi]| \\
&\leq |\rho^2[\Phi] - \rho^2[\Psi]| \cdot |(\beta_a\beta_b\beta_c)^{\frac{1}{3}}[\Phi]| \\
&+ |\rho^2[\Psi]| \cdot |(\beta_a\beta_b\beta_c)^{\frac{1}{3}}[\Phi] - (\beta_a\beta_b\beta_c)^{\frac{1}{3}}[\Psi]| \\
&\leq |(\beta_a\beta_b\beta_c)^{\frac{1}{3}}[\Phi]| d_{\mathcal{E}}(\Phi, \Psi) \\
&+ \left(\rho^2[\Psi] \max\{|(\beta_a\beta_b\beta_c)^{\frac{1}{3}}[\Phi]|, |(\beta_a\beta_b\beta_c)^{\frac{1}{3}}[\Psi]|\} \right) \times \\
&\times \left(\left| \frac{\xi_a[\Phi] + \xi_b[\Phi] + \xi_c[\Phi] - \xi_a[\Psi] - \xi_b[\Psi] - \xi_c[\Psi]}{3\mathbf{h}} \right| \right) \\
&\leq 2M d_{\mathcal{E}}(\Phi, \Psi) + \frac{2}{3\mathbf{h}} \rho^2[\Psi] M(d_{\mathcal{E}}(\Phi, \Psi) \\
&+ |\xi_{a,c}[\Phi] - \xi_{a,c}[\Psi] + \xi_a[\Psi] - \xi_a[\Phi]|) \\
&\leq 2M d_{\mathcal{E}}(\Phi, \Psi) + \frac{2}{3\mathbf{h}} \rho^2[\Psi] M(2d_{\mathcal{E}}(\Phi, \Psi) + |\xi_a[\Phi] - \xi_a[\Psi]|) \\
&\leq 2M d_{\mathcal{E}}(\Phi, \Psi) + \frac{2}{3\mathbf{h}} M_2 M(2d_{\mathcal{E}}(\Phi, \Psi) + 2^3 \mathbf{h} M_1 d_{\mathcal{E}}(\Phi, \Psi)) \\
&\leq 2M M_1 d_{\mathcal{E}}(\Phi, \Psi) + 2^3 M M_1 M_2 d_{\mathcal{E}}(\Phi, \Psi) \\
&\leq 2(1 + 2^2 M_2) M M_1 d_{\mathcal{E}}(\Phi, \Psi) \\
&\leq 2^4 M M_1 M_2 d_{\mathcal{E}}(\Phi, \Psi)
\end{aligned}$$

The estimates above imply (using $|\beta_p^2[\Phi] - \beta_p^2[\Psi]| \leq 2\max\{|\beta_p[\Phi]|, |\beta_p[\Psi]|\}|\beta_p[\Phi] - \beta_p[\Psi]|$):

$$\begin{aligned}
|\mathbf{a}_b[\Phi, \mathbf{h}, Z, B_a] - \mathbf{a}_b[\Psi, \mathbf{h}, Z, B_a]| &\leq |\beta_b^2[\Phi] - \beta_b^2[\Psi]| + |\beta_c^2[\Phi] - \beta_c^2[\Psi]| + 2|\beta_a[\Phi]\beta_c[\Phi] - \beta_a[\Psi]\beta_c[\Psi]| \\
&+ \frac{2}{3}|\gamma^2[\Phi] - \gamma^2[\Psi]| \\
&\leq 2^{10} M M_1 M_2 d_{\mathcal{E}}(\Phi, \Psi) + 2^5 M M_1 M_2 d_{\mathcal{E}}(\Phi, \Psi) + \frac{2^5}{3} M M_1 M_2 d_{\mathcal{E}}(\Phi, \Psi) \\
&\leq 2^{11} M M_1 M_2 d_{\mathcal{E}}(\Phi, \Psi)
\end{aligned}$$

Analogous estimate holds for $|\mathbf{a}_c[\Phi, \mathbf{h}, Z, B_a] - \mathbf{a}_c[\Psi, \mathbf{h}, Z, B_a]|$.

Further,

$$\begin{aligned}
|\beta_p[\Phi]\beta_q[\Phi] - \beta_p[\Psi]\beta_q[\Psi]| &\leq |\beta_q[\Psi]| \cdot |\beta_p[\Phi] - \beta_p[\Psi]| + |\beta_p[\Phi]| \cdot |\beta_q[\Phi] - \beta_q[\Psi]| \\
&\leq 2^2 \sqrt{MM_2} \cdot 2^6 M_1 \sqrt{MM_2} d_{\mathcal{E}}(\Phi, \Psi) + 2^2 \sqrt{MM_2} \cdot 2^6 M_1 \sqrt{MM_2} d_{\mathcal{E}}(\Phi, \Psi) \\
&\leq 2^9 MM_1 M_2 d_{\mathcal{E}}(\Phi, \Psi)
\end{aligned}$$

and therefore

$$\begin{aligned}
|\mathfrak{a}_a[\Phi, \mathbf{h}, Z, B_a] - \mathfrak{a}_a[\Psi, \mathbf{h}, Z, B_a]| &\leq |\beta_b^2[\Phi] - \beta_b^2[\Psi]| + |\beta_c^2[\Phi] - \beta_c^2[\Psi]| + 2|\beta_b[\Phi]\beta_c[\Phi] - \beta_b[\Psi]\beta_c[\Psi]| \\
&\quad + \frac{2}{3}|\gamma^2[\Phi] - \gamma^2[\Psi]| \\
&\leq (2^3 \sqrt{MM_2} \cdot 2^6 M_1 \sqrt{MM_2} d_{\mathcal{E}}(\Phi, \Psi)) \cdot 2 + 2 \cdot 2^9 MM_1 M_2 d_{\mathcal{E}}(\Phi, \Psi) \\
&\quad + \frac{2}{3} 2^4 MM_1 M_2 d_{\mathcal{E}}(\Phi, \Psi) \\
&\leq 2^{12} MM_1 M_2 d_{\mathcal{E}}(\Phi, \Psi).
\end{aligned}$$

These estimates imply

$$\begin{aligned}
|I_1[\Phi] - I_1[\Psi]| &= \left| -\frac{1}{\mathbf{h}} \mathfrak{a}_a[\Phi, \mathbf{h}, Z, B_a] \tanh \phi_a[\Phi] + \frac{1}{\mathbf{h}} \mathfrak{a}_a[\Psi, \mathbf{h}, Z, B_a] \tanh \phi_a[\Psi] \right| \\
&\leq M_1 \left| -\mathfrak{a}_a[\Phi, \mathbf{h}, Z, B_a] \tanh \phi_a[\Phi] + \mathfrak{a}_a[\Psi, \mathbf{h}, Z, B_a] \tanh \phi_a[\Psi] \right| \\
&\leq M_1 |\mathfrak{a}_a[\Phi, \mathbf{h}, Z, B_a]| |\tanh \phi_a[\Phi] - \tanh \phi_a[\Psi]| + M_1 |\mathfrak{a}_a[\Phi, \mathbf{h}, Z, B_a] - \mathfrak{a}_a[\Psi, \mathbf{h}, Z, B_a]| \\
&\leq 2^{13} MM_1^2 M_2 d_{\mathcal{E}}(\Phi, \Psi)
\end{aligned}$$

and

$$\begin{aligned}
|I_{(3,p)}[\Phi] - I_{(3,p)}[\Psi]| &= \left| \frac{1}{\mathbf{h}} \mathfrak{a}_p[\Phi, \mathbf{h}, Z, B_a] + \frac{1}{\mathbf{h}} \mathfrak{a}_a[\Phi, \mathbf{h}, Z, B_a] - \frac{1}{\mathbf{h}} \mathfrak{a}_p[\Psi, \mathbf{h}, Z, B_a] - \frac{1}{\mathbf{h}} \mathfrak{a}_a[\Psi, \mathbf{h}, Z, B_a] \right| \\
&\leq M_1 |\mathfrak{a}_p[\Phi, \mathbf{h}, Z, B_a] - \mathfrak{a}_p[\Psi, \mathbf{h}, Z, B_a]| + M_1 |\mathfrak{a}_a[\Phi, \mathbf{h}, Z, B_a] - \mathfrak{a}_a[\Psi, \mathbf{h}, Z, B_a]| \\
&\leq 2^{13} MM_1^2 M_2 d_{\mathcal{E}}(\Phi, \Psi)
\end{aligned}$$

and

$$\begin{aligned}
|I_2[\Phi] - I_2[\Psi]| &\leq \left| \frac{1}{A_a^2[\Phi]} \mathfrak{a}_a[\Phi, \mathbf{h}, Z, B_a] \right| \cdot |\phi_a[\Phi] \tanh \phi_a[\Phi] - \phi_a[\Psi] \tanh \phi_a[\Psi]| \\
&\quad + \left| \frac{1}{A_a^2[\Phi]} \mathfrak{a}_a[\Phi, \mathbf{h}, Z, B_a] - \frac{1}{A_a^2[\Psi]} \mathfrak{a}_a[\Psi, \mathbf{h}, Z, B_a] \right| \cdot |1 - \phi_a[\Psi] \tanh \phi_a[\Psi]| \\
&\leq 2^{18} MM_1^2 M_2 d_{\mathcal{E}}(\Phi, \Psi)
\end{aligned}$$

The last step followed from (using $|\frac{1}{x^2} - \frac{1}{y^2}| \leq \frac{1}{|x^2 y^2|} 2 \max\{|x|, |y|\} |x - y|$):

$$\begin{aligned} |\phi_a[\Phi] \tanh \phi_a[\Phi] - \phi_a[\Psi] \tanh \phi_a[\Psi]| &\leq |\phi_a[\Phi]| |\phi_a[\Phi] - \phi_a[\Psi]| + |\tanh \phi_a[\Psi]| |\phi_a[\Phi] - \phi_a[\Psi]| \\ &\leq (2^2 + 1) 2^2 M_1^2 d_{\mathcal{E}}(\Phi, \Psi) \\ &\leq 2^5 M_1^2 d_{\mathcal{E}}(\Phi, \Psi) \end{aligned}$$

and

$$\begin{aligned} \left| \frac{1}{A_a^2[\Phi]} \mathbf{a}_a[\Phi, \mathbf{h}, Z, B_a] - \frac{1}{A_a^2[\Psi]} \mathbf{a}_a[\Psi, \mathbf{h}, Z, B_a] \right| &\leq \left| \frac{1}{A_a^2[\Phi]} (\mathbf{a}_a[\Phi, \mathbf{h}, Z, B_a] - \mathbf{a}_a[\Psi, \mathbf{h}, Z, B_a]) \right| \\ &\quad + \left| \mathbf{a}_a[\Psi, \mathbf{h}, Z, B_a] \left(\frac{1}{A_a^2[\Phi]} - \frac{1}{A_a^2[\Psi]} \right) \right| \\ &\leq 2^2 |\mathbf{a}_a[\Phi, \mathbf{h}, Z, B_a] - \mathbf{a}_a[\Psi, \mathbf{h}, Z, B_a]| \\ &\quad + 2^{11} M M_2 d_{\mathcal{E}}(\Phi, \Psi) \\ &\leq 2^{14} M M_1 M_2 d_{\mathcal{E}}(\Phi, \Psi) \end{aligned}$$

and

$$\begin{aligned} |1 - \phi_a[\Psi] \tanh \phi_a[\Psi]| &\leq 1 + |\phi_a[\Psi]| |\tanh \phi_a[\Psi]| \\ &\leq 2^3 M_1 \end{aligned}$$

Definition 3.12. (*State Vectors*). $\forall \pi = (a, b, c) \in S_3$, $\mathbf{f} = (\mathbf{h}, w, q, z) \in (0, \infty)^2 \times \mathbb{R} \times (0, \infty)$, $\sigma_* \in \{-1, +1\}^3$, the field $\Phi_* = \Phi_*(\pi, \mathbf{f}, \sigma_*)$ is given by

$$\begin{aligned} \alpha_a[\Phi_*] &= -1 \\ \alpha_b[\Phi_*] &= \frac{w}{1+w} \\ \alpha_c[\Phi_*] &= -w - 4(1+w)\gamma^2 - \mu \\ \beta_a[\Phi_*, \mathbf{h}] &= -\frac{1+w}{1+2w}(1 + \mathbf{h} \log 2) \\ \beta_b[\Phi_*, \mathbf{h}] &= -\frac{1+w}{1+2w}(1 + \mathbf{h} \log 2) \\ \beta_c[\Phi_*, \mathbf{h}] &= -(1+w)q - \frac{w(1+w)}{1+2w} - \frac{1+3w+w^2}{1+2w} \mathbf{h} \log 2 \end{aligned}$$

where $\mu = (1+w)(\beta_1^2 + \beta_2^2 + \beta_3^2 - 2\beta_2\beta_3 - 2\beta_3\beta_1 - 2\beta_1\beta_2)$ is determined uniquely by requiring that the new constraint, i.e. new equation (1.1c) holds (see end of Section 1).

$$\rho[\Phi_*] = \rho[\Phi_0] = \sqrt{z}.$$

Definition 3.13. $\forall \pi = (a, b, c) \in S_3$, $\sigma_* \in \{+1, -1\}^3$ let $\mathcal{H}(\pi, \sigma_*) \subset \mathcal{D}(\sigma_*)$ be the set of all vectors $\Phi = \alpha \oplus \beta \oplus \gamma \in \mathcal{D}(\sigma_*)$ with

$$(37) \quad |\beta_a| = |\beta_b| \quad \sum_{(i,j,k) \in \mathcal{C}} (\alpha_j \alpha_k - (\beta_i)^2 + 2\beta_j \beta_k) - 4\gamma^2 = 0$$

$$(38) \quad 0 < \alpha_b < -\alpha_a \quad (\alpha_b + |\alpha_a|) \log |\beta_a / \alpha_a| < \alpha_b \log 2$$

Lemma 3.7. Let $\pi = (a, b, c) \in S_3$ and $\sigma_* \in \{+1, -1\}^3$. The set $\mathcal{H}(\pi, \sigma_*) \subset \mathcal{D}(\sigma_*)$ is a smooth 4-dim submanifold. The map

$$(0, \infty)^3 \times \mathbb{R} \times (0, \infty) \rightarrow \mathcal{H}(\pi, \sigma_*)$$

$$(39) \quad (\lambda, \mathbf{h}, w, q, z) \mapsto \lambda \Phi_*(\pi, (\mathbf{h}, w, q, z), \sigma_*)$$

is a diffeomorphism. Its inverse is given by

$$(40) \quad w = -\frac{\alpha_b}{\alpha_a + \alpha_b}$$

$$(41) \quad \lambda = -\alpha_a$$

$$(42) \quad \frac{1}{\mathbf{h}} = -\frac{1+2w}{1+w} \log |\beta_a / \alpha_a| + \frac{w}{1+w} \log 2$$

$$(43) \quad q = -\frac{1}{1+w} \mathbf{h} \log |\beta_c / \alpha_a| - \frac{w}{1+2w} (1 + \mathbf{h} \log 2)$$

$$(44) \quad z = -\frac{\rho^2}{\alpha_a}$$

Note: z is well-defined by (38).

Definition 3.14. $\forall \mathbf{f} = (\mathbf{h}, w, q, z) \in (0, \infty)^4$ set

$$\begin{aligned} \tau_{1-}(\mathbf{f}) &= \begin{cases} -\frac{1+w}{3+w} q - \frac{1}{3+w} \mathbf{h} \log 2 & \text{if } q \leq 1 \\ -\frac{1+w}{3+2w} - \frac{1+w}{3+2w} \mathbf{h} \log 2 & \text{if } q > 1 \end{cases} \\ \tau_{1+}(\mathbf{f}) &= (1 + \mathbf{h} \log 2) \frac{1+w}{1+2w} \end{aligned}$$

Note that $\tau_{1-}(\mathbf{f}) < 0$ and $\tau_{1+}(\mathbf{f}) > 0$.

Motivation for Definition 3.14. It gives us the approximate time of crossing for β 's. For example, if β_a bounces at $\tau = 0$, then β_b is going to make the next bounce (to the right) at $\tau_{1+} > 0$.

Definition 3.15. (Initial Data) $\forall \pi = (a, b, c) \in S_3$, $\forall \mathbf{f} = (\mathbf{h}, w, q, z) \in (0, \infty)^4$, $\forall \sigma_* \in \{-1, +1\}^3$ let

$\Phi_1 = \Phi_1(\pi, \mathbf{f}, \sigma_*) : \mathbb{R} \rightarrow \mathcal{D}(\sigma_*)$ be given by

$$\begin{aligned} A_a[\Phi_1](\tau) &= A_a[\Phi_*], \quad \theta_a[\Phi_1, \mathbf{h}](\tau) = \theta_a[\Phi_*, \mathbf{h}] \\ \alpha_{p,a}[\Phi_1](\tau) &= \alpha_{p,a}[\Phi_*], \quad \xi_{p,a}[\Phi_1, \mathbf{h}](\tau) = \xi_{p,a}[\Phi_*, \mathbf{h}] + (\tau - \tau_{1+})\alpha_{p,a}[\Phi_*] \\ \rho[\Phi_1] &= \rho[\Phi_*] = \sqrt{z} \end{aligned}$$

for all $\tau \in \mathbb{R}$ and $p \in \{b, c\}$. Here, $\tau_{1+} = \tau_{1+}(\mathbf{f})$ and $\Phi_* = \Phi_*(\pi, \mathbf{f}, \sigma_*)$.

Lemma 3.8. *Stays the same as in [2] since $\rho[\Phi_0] = \rho[\Phi_1] = \sqrt{z}$. That is, $\forall \pi \in (a, b, c) \in S_3$, $\mathbf{f} = (\mathbf{h}, w, q, z) \in (0, \infty)^4$, $\sigma_* \in \{-1, +1\}^3$, set $\Phi_0 = \Phi_0(\pi, \mathbf{f}, \sigma_*)$, $\tau_{1+} = \tau_{1+}(\mathbf{f})$. Then*

- (a) $|\beta_a[\Phi_1](\tau_{1+})| = |\beta_b[\Phi_1](\tau_{1+})|$
- (b) $c[\Phi_1, \mathbf{h}, Z](\tau_{1+}) = 0$
- (c) $d_{\mathcal{D}}(\Phi_0(\tau_{1+}), \Phi_1(\tau_{1+})) \leq 2^7 \max\{1 + w, \mathbf{h}\} \exp\left(-\frac{1}{2\mathbf{h}} \min\{1, w + q\}\right)$
- (d) $d_{\mathcal{D}}(\Phi_0(\tau), \Phi_1(\tau)) \leq (1 + |\tau - \tau_{1+}|)d_{\mathcal{D}}(\Phi_0(\tau_{1+}), \Phi_1(\tau_{1+})), \forall \tau \in \mathbb{R}$

Definition 3.16. (Approximate Transfer Maps) *Introduce three maps*

$$\begin{aligned} \mathcal{P}_L : S_3 \times (0, \infty)^4 &\rightarrow S_3 \\ \mathcal{Q}_L : (0, \infty)^4 &\rightarrow (0, \infty)^2 \times \mathbb{R} \times (0, \infty) \\ \lambda_L : (0, \infty)^4 &\rightarrow (0, \infty) \end{aligned}$$

where $\mathbf{f} = (\mathbf{h}, w, q, z)$, $q_L = \text{num}1_L / \text{den}_L$, $\mathbf{h}_L = \text{num}2_L / \text{den}_L$, and:

- if $q \leq 1$:

$$\begin{aligned} (a', b', c') &= (c, a, b) \\ w_L &= \frac{1}{1 + w} \\ \lambda_L &= 2 + w \\ \text{num}1_L &= (1 + w)(1 - q) - \mathbf{h} \log 2 + \mathbf{h} w \log(2 + w) \\ \text{num}2_L &= \mathbf{h}(2 + w) \\ \text{den}_L &= (1 + w)(q - \mathbf{h} \log 2) + \mathbf{h}(3 + w) \log(2 + w) \\ z_L &= \frac{1}{2 + w} z \end{aligned}$$

- if $q > 1$

$$\begin{aligned}
(a', b', c') &= (b, a, c) \\
w_L &= 1 + w \\
\lambda_L &= \frac{2 + w}{1 + w} \\
\text{num}1_L &= (1 + w)(q - 1 - \mathbf{h} \log 2) - \mathbf{h} w \log \frac{2 + w}{1 + w} \\
\text{num}2_L &= \mathbf{h}(2 + w) \\
\text{den}_L &= (1 + w) - \mathbf{h} \log 2 + \mathbf{h}(3 + 2w) \log \frac{2 + w}{1 + w} \\
z_L &= \frac{1 + w}{2 + w} z
\end{aligned}$$

Motivation for Definition 3.16

Introduce the scaling parameter λ_L . Recall $\rho[\Phi_0] = \rho[\Phi_*] = \sqrt{z}$. We have

$$(45) \quad \Phi_0(\pi, f, \sigma) \big|_{\tau_1-} = \lambda_L \Phi_*(\pi', f_L, \sigma),$$

with $f = (\mathbf{h}, w, q, z)$, $\pi = (a, b, c)$ and $\pi' = (a', b', c')$. Therefore, the rescaling for \sqrt{z} will be $\lambda_L \cdot \lambda_L^{-\frac{1}{6} \cdot 3} = \sqrt{\lambda_L}$. To get the $f_L = (\mathbf{h}_L, w_L, q_L, z_L)$ we calculate:

Case1: $q > 1$. Then $b' = a$, $a' = b$ and (45) is equivalent to

$$\begin{aligned}
\alpha_a[\Phi_0] &= \lambda_L \alpha_{b'}[\Phi_*](\mathbf{h}_L, w_L, q_L) \\
\alpha_b[\Phi_0] &= \lambda_L \alpha_{a'}[\Phi_*](\mathbf{h}_L, w_L, q_L) \\
\frac{1}{\mathbf{h}} \xi_a[\Phi_0] &= \frac{1}{\mathbf{h}_L} \xi_{b'}[\Phi_*](\mathbf{h}_L, w_L, q_L) + \log \lambda_L \\
\frac{1}{\mathbf{h}} \xi_b[\Phi_0] &= \frac{1}{\mathbf{h}_L} \xi_{a'}[\Phi_*](\mathbf{h}_L, w_L, q_L) + \log \lambda_L \\
\rho[\Phi_0] &= \sqrt{\lambda_L} \rho[\Phi_*]
\end{aligned}$$

Solving for f_L we get the same expressions for \mathbf{h}_L , w_L and q_L as in the vacuum case and the energy density for the fluid goes as $z_L = \frac{1+w}{2+w} z$.

Case2: $q \leq 1$. Then $b' = a$, $c' = b$ and (45) is equivalent to

$$\begin{aligned}
\alpha_a[\Phi_0] &= \lambda_L \alpha_{b'}[\Phi_*](\mathbf{h}_L, w_L, q_L) \\
\alpha_b[\Phi_0] &= \lambda_L \alpha_{c'}[\Phi_*](\mathbf{h}_L, w_L, q_L) \\
\frac{1}{\mathbf{h}} \xi_a[\Phi_0] &= \frac{1}{\mathbf{h}_L} \xi_{b'}[\Phi_*](\mathbf{h}_L, w_L, q_L) + \log \lambda_L \\
\frac{1}{\mathbf{h}} \xi_b[\Phi_0] &= \frac{1}{\mathbf{h}_L} \xi_{c'}[\Phi_*](\mathbf{h}_L, w_L, q_L) + \log \lambda_L \\
\rho[\Phi_0] &= \sqrt{\lambda_L} \rho[\Phi_*]
\end{aligned}$$

for which we get $z_L = \frac{1}{2+w}z$.

Lemma 3.9. *The identities for λ_L , w_L , \mathbf{h}_L and q_L stay the same as in [2] (verified by direct substitution using Definition 3.14). Additionally, we have*

$$(46) \quad \frac{z}{z_L} = 1 - \alpha_{a,a'}[\Phi_0](\tau_{1-}) = 1 - \alpha_{a,a'}[\Phi_0](\tau)$$

with $\Phi_0 = \Phi_0(\pi, \mathbf{f}, \sigma_*)$ and $\tau \in \mathbb{R}$.

Definition 3.17. $\forall \mathbf{f} = (\mathbf{h}, w, q, z) \in (0, \infty)^4$ define

$$(47) \quad \tau_*(\mathbf{f}) = \begin{cases} \frac{q}{1+w}, & \text{if } q \leq 1 \\ 1, & \text{if } q > 1. \end{cases}$$

Definition 3.18. $\forall \mathbf{f} = (\mathbf{h}, w, q, z) \in (0, \infty)^4$ set

$$(48) \quad \mathbf{K}(\mathbf{f}) = 2^{40} \left(\frac{1}{\mathbf{h}} \right)^2 \max\left\{ \left(\frac{1}{w} \right)^2, w^3 \right\} \max\left\{ \left(\frac{1}{q} \right)^2, q \right\} \exp\left(-\frac{1}{\mathbf{h}} 2^{-7} \tau_*(\mathbf{f}) \right) \max\{1, z\}^2$$

Definition 3.19. Let \mathcal{F} be the open set of all $\mathbf{f} = (\mathbf{h}, w, q, z) \in (0, \infty)^4$ for which $q \neq 1$, $\mathbf{K}(\mathbf{f}) < 1$, $\mathbf{h} < 2^{-7} \tau_*(\mathbf{f})$.

Proposition 3.3. $\pi = (a, b, c) \in S_3$, $\sigma_* \in \{-1, +1\}^3$, $\mathbf{f} = (\mathbf{h}, w, q, z)$. There are *unique* maps

$$\begin{aligned} \Pi &= \Pi[\pi, \sigma_*] : \mathcal{F} \rightarrow (0, \infty)^2 \times \mathbb{R} \times (0, \infty) \\ \Lambda &= \Lambda[\pi, \sigma_*] : \mathcal{F} \rightarrow [1, \infty) \\ \tau_{2-} &= \tau_{2-}[\pi, \sigma_*] : \mathcal{F} \rightarrow (-\infty, 0) \end{aligned}$$

so that $\forall \mathbf{f} = (\mathbf{h}, w, q, z) \in \mathcal{F}$ we have

- (1) $\|\Pi(\mathbf{f}) - Q_L(\mathbf{f})\|_{\mathbb{R}^3} \leq \mathbf{K}(\mathbf{f})$
- (2) $|\Lambda(\mathbf{f}) - \lambda_L(\mathbf{f})| \leq \mathbf{K}(\mathbf{f})$
- (3) $\tau_-(\mathbf{f}) < \tau_{2-}(\mathbf{f}) < \frac{1}{2}\tau_{1-}(\mathbf{f})$ and $|\tau_{2-}(\mathbf{f}) - \tau_{1-}(\mathbf{f})| \leq \mathbf{K}(\mathbf{f})$
- (4) Π, Λ and τ_{2-} are continuous
- (5) if we set $\tau_{2-} = \tau_{2-}(\mathbf{f})$, $\tau_{2+} = \tau_{1+}(\mathbf{f})$, $\pi' = (a', b', c') = \mathcal{P}_L(\pi, \mathbf{f})$, $\lambda = \Lambda(\mathbf{f})$ and $\mathbf{f}' = (\mathbf{h}', w', q', z') = \Pi(\mathbf{f})$, then $\frac{1}{2} \leq \tau_{2+} - \tau_{2-} \leq 3$ and there is a smooth field

$$\Phi = \alpha \oplus \beta \oplus \gamma \in \mathcal{E} = \mathcal{E}(\sigma_*; \tau_{2-}, \tau_{2+})$$

that satisfies

- $(\mathbf{a}, \mathbf{b}, c, \mathbf{d})[\Phi, \mathbf{h}, Z] = 0$ on $[\tau_{2-}, \tau_{2+}]$ (i.e. solution to the primary system of equations in Section 1)
- $\Phi(\tau_{2+}) = \Phi_*(\pi, \mathbf{f}, \sigma_*)$ and $\Phi(\tau_{2-}) = \lambda \Phi_*(\pi', \mathbf{f}', \sigma_*)$, in particular $\Phi(\tau_{2+}) \in \mathcal{H}(\pi, \sigma_*)$ and $\Phi(\tau_{2-}) \in \mathcal{H}(\pi', \sigma_*)$

- $|\beta_a[\Phi](\tau)| \geq |\beta_{a'}[\Phi](\tau)|$ for all $\tau \in [\tau_{2-}, \frac{1}{2}\tau_{1-}(\mathbf{f})]$ with equality if and only if $\tau = \tau_{2-}$
- $d_{\mathcal{E},(\pi,\mathbf{h})}(\Phi, \Phi_0) \leq \mathbf{K}(\mathbf{f})$, where $\Phi_0 = \Phi_0(\pi, \mathbf{f}, \sigma_*)|_{[\tau_{2-}, \tau_{2+}]}$
- $\sup_{\tau \in [\tau_{2-}, \tau_{2+}]} \max\{\alpha_{b,c}[\Phi], \alpha_{c,a}[\Phi], \alpha_{a,b}[\Phi]\}(\tau) \leq -2^{-2} \min\{w^2, w^{-1}\}$

Proof. The logic of the proof is the same as for the corresponding proposition in [2]. We start by introducing the parameter vector $\ell = (\ell_1, \dots, \ell_8) \in \mathbb{R}^8$.

$\pi = (a, b, c) \in S_3$, $\sigma_*\{-1, +1\}^3$, $\mathbf{f} = (\mathbf{h}, w, q, z) \in (0, \infty)^4$, $q \neq 1$, $\mathbf{h} \leq 1$.

For any $s = (s_1, \dots, s_8) \in \mathbb{R}^8$ define

$$\begin{aligned} \mathbf{X}(s) &= \mathbf{X}(s_1, \dots, s_8) \\ &= 2^{s_1} \left(\frac{1}{\mathbf{h}}\right)^{s_2} \times \begin{cases} \frac{1}{w}^{s_3} & \text{if } w \leq 1 \\ w^{s_4} & \text{if } w > 1 \end{cases} \times \begin{cases} \frac{1}{q}^{s_5} & \text{if } q \leq 1 \\ q^{s_6} & \text{if } q > 1 \end{cases} \times \exp\left(\frac{1}{\mathbf{h}} s_7 \tau_*\right) \times \max\{1, z\}^{s_8} \end{aligned}$$

Properties of $\mathbf{X}(s)$ (stay unchanged):

- $\mathbf{X}(s)\mathbf{X}(s') = \mathbf{X}(s + s')$;
- $\mathbf{X}(0, \dots, 0) = 1$;
- $\mathbf{X}(-1, 0, 0, -1, -1, 0, 0, 0) = \frac{q}{2w} \leq \tau_*$;
- $\mathbf{X}(s)$ is positive, non-decreasing in each argument.

Basic smallness assumptions. Introduce parameter vector $\ell = (\ell_1, \dots, \ell_9) \in \mathbb{R}^9$ with $(\ell_1, \dots, \ell_8) \geq (0, 0, 0, 0, 0, 0, -\infty, 0)$ and $\ell_9 \geq 0$. $(\bullet)_1$

Basic assumptions on $\mathbf{f} = (\mathbf{h}, w, q, z)$ are

$$(49) \quad q \neq 1 \quad \mathbf{K} := \mathbf{X}(\ell) < 1 \quad \mathbf{h} < 2^{-\ell_9} \tau_*$$

We are interested in the ratio \mathbf{h}/τ_* . The component of ℓ giving a bound for it is ℓ_9 .

Abbreviations. See [2].

Preliminaries 1. Introduce ϵ_- and ϵ_+ by $\tau_{0-} = \tau_- + \epsilon_-$ and $\tau_{0+} = \tau_+ - \epsilon_+$. Recall Lemma 3.4. By using the definitions for τ_+ , τ_{1+} , τ_- and τ_{1-} , we find

$$\begin{aligned} \epsilon_+ &= \tau_+ - \tau_{0+} = \tau_+ - \tau_{1+} = \frac{1+w}{1+2w} \left(1 + \frac{1}{w} - \mathbf{h} \log 2\right) \\ \epsilon_- &= \tau_{0-} - \tau_- = \frac{1}{2}(\tau_{1-} - \tau_-) = \begin{cases} \frac{1}{2(3+w)} \left(\frac{1+w}{2+w} q - \mathbf{h} \log 2\right) & \text{if } q < 1 \\ \frac{1+w}{2(3+2w)} \left(\frac{1+w}{2+w} - \mathbf{h} \log 2\right) & \text{if } q > 1. \end{cases} \end{aligned}$$

Require $\ell_9 \geq 2 (\bullet)_2$. Then $\mathbf{h} \log 2 \leq \mathbf{h}$ and $\mathbf{h} \stackrel{(49), (\bullet)_2}{<} 2^{-2} \tau_* \stackrel{(47)}{\leq} 2^{-2} \min\{1, q\}$. Recall $\tau_+ = 1 + \frac{1}{w}$. Then we have

$$\begin{aligned} \frac{\epsilon_+}{\tau_+} &= \frac{1+w}{1+2w} \left(1 + \frac{1}{w} - \mathbf{h} \log 2 \right) \cdot \frac{w}{1+w} \\ &= \frac{1+w}{1+2w} - \frac{w}{1+2w} \mathbf{h} \log 2 \\ &\stackrel{(\bullet)_2}{\geq} \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{4} = \frac{3}{8} > 2^{-2} \end{aligned}$$

On the other hand, $\frac{\epsilon_+}{\tau_+} \leq \frac{1+w}{1+2w} \leq 1$. Therefore,

$$(50) \quad 2^{-2} \leq \frac{\epsilon_+}{\tau_+} \leq 1$$

Analogous calculations (for $q < 1$, $\frac{\epsilon_-}{\tau_*} = \frac{1+w}{q} \frac{1}{2(3+w)} \left(\frac{1+w}{2+w} q - \mathbf{h} \log 2 \right) \leq \frac{(1+w)^2}{2(1+w)(2+w)} \leq 2^{-1}$, etc.) lead to

$$(51) \quad 2^{-5} \leq \frac{\epsilon_-}{\tau_*} \leq 2^{-1}$$

We have

$$(52) \quad -1 < \tau_- < \tau_{0-} < \tau_{1-} < 0 < \frac{1}{2} < \tau_{0+} = \tau_{1+} = \tau_{2+} < \min\{2, \tau_+\}$$

Define $\delta := 2^{-9} \min\{1, w\} \tau_*$. Recall *Properties of $\mathbf{X}(s)$* . Then we have

$$\begin{aligned} \delta &= \mathbf{X}(-9, 0, -1, 0, 0, 0, 0, 0) \tau_* \\ &\geq \mathbf{X}(-9, 0, -1, 0, 0, 0, 0, 0) \mathbf{X}(-1, 0, 0, -1, -1, 0, 0, 0) \\ (53) \quad &= \mathbf{X}(-10, 0, -1, -1, -1, 0, 0, 0) \end{aligned}$$

Therefore, from (51) follows that $\tau_* \leq 2^5 \epsilon_-$ and $\delta \leq 2^{-4} \min\{1, w, \epsilon_-, \frac{\epsilon_+}{\tau_+ \tau_{0+}}\}$. This implies that the assumption of Lemma 3.4 is satisfied.

Preliminaries 2. Require $\ell_8 \geq 7 (\bullet)_3$. Recall Lemmas 3.2 and 3.8. Then we have

$$\begin{aligned}
d_{\mathcal{E}}(\Phi_0, \Phi_1) &\stackrel{\text{Lemma 3.8, d)}}{\leq} 2^2 d_{\mathcal{D}}(\Phi_0(\tau_{1+}), \Phi_*) \\
&\stackrel{\text{Lemma 3.2, b) and } (\bullet)_3}{\leq} 2^{11} d_{\mathcal{D}}(\Phi_0(\tau_{1+}), \Phi_*) \\
&\stackrel{\text{Lemma 3.8, c)}}{\leq} 2^{18} (1+w) \exp\left(-\frac{1}{2\mathbf{h}} \min\{1, q\}\right) \\
&\stackrel{(47)}{\leq} 2^{18} (1+w) \exp\left(-\frac{1}{2\mathbf{h}} \tau_*$$

Note on the second inequality: the assumption $(\bullet)_3$ plays the crucial role here. In particular, we need to make sure that the assumptions of Lemma 3.2 are satisfied, that is,

- $\mathbf{h} \leq 1 \rightarrow$ ok by assumption $(\bullet)_3$;
- constants $C, D \geq 1$ such that $C^{-1} \leq A_a[X] \leq C$ and $D^{-1} \leq \mathbf{h}|\phi_a[X]| \leq D \rightarrow$ ok by putting $C = D = 2$ (recall $A_a[\Phi_0](\tau_{1+}) = 1$ and $\mathbf{h}\phi_a[\Phi_0](\tau_{1+}) = \tau_{1+} \in [\frac{1}{2}, 2]$, by Definition 3.14). We want to use statement b) from Lemma 3.2. Indeed, $\exp(-\frac{1}{\mathbf{h}} 2^{-2} 2^{-1}) \leq 2^{-12}$ is satisfied for l_8 at least 7. That is the motivation for the assumption $(\bullet)_3$.

Recall the estimate for $d_{\mathcal{E}}(\Phi_0, \Phi_1)$. Therefore, the assumption $(\ell_1, \dots, \ell_8) \geq (31, 0, 1, 2, 1, 0, -2^{-1}, 0)$ $(\bullet)_4$ implies immediately that

$$(54) \quad \Phi_1 \in B_{\mathcal{E}}[2^{-2}\delta, \Phi_0]$$

Construction of Φ . Fixed point construction is technically carried out on the interval $[\tau_{0-}, \tau_{0+}]$. Define a map $P : B_{\mathcal{E}}[\delta, \Phi_0] \rightarrow B_{\mathcal{E}}[\delta, \Phi_0], \Psi \mapsto P(\Psi)$ by

$$(55) \quad A_a[P(\Psi)](\tau) - A_a[\Phi_1](\tau) = \int_{\tau_{0+}}^{\tau} d\tau' \mathbf{I}_1[\Psi, \mathbf{h}, \pi](\tau')$$

$$(56) \quad \theta_a[P(\Psi), \mathbf{h}](\tau) - \theta_a[\Phi_1, \mathbf{h}](\tau) = \int_{\tau_{0+}}^{\tau} d\tau' \mathbf{I}_2[\Psi, \mathbf{h}, \pi](\tau')$$

$$(57) \quad \alpha_{p,a}[P(\Psi)](\tau) - \alpha_{p,a}[\Phi_1](\tau) = \int_{\tau_{0+}}^{\tau} d\tau' \mathbf{I}_{(3,p)}[\Psi, \mathbf{h}, \pi](\tau')$$

$$(58) \quad \xi_{p,a}[P(\Psi), \mathbf{h}](\tau) - \xi_{p,a}[\Phi_1, \mathbf{h}](\tau) = \int_{\tau_{0+}}^{\tau} d\tau'' \int_{\tau_{0+}}^{\tau''} d\tau' \mathbf{I}_{(3,p)}[\Psi, \mathbf{h}, \pi](\tau')$$

$$(59) \quad \rho[P(\Psi)](\tau) - \rho[\Phi_1](\tau) = 0$$

for all $p \in \{b, c\}$ and $\tau \in [\tau_{0-}, \tau_{0+}]$. We now want to make sure that P is well defined. Let $\Psi \in B_{\mathcal{E}}[\delta, \Phi_0]$. The requirement $(\ell_1, \dots, \ell_8) \geq (29, 2, 1, 1, 1, 0, -2^{-7}, 1)$ $(\bullet)_5$ makes sure that it's the case, since we have

$$\begin{aligned} d_{\mathcal{E}}(P(\Psi), \Phi_0) &\leq d_{\mathcal{E}}(P(\Psi), \Phi_1) + d_{\mathcal{E}}(\Phi_1, \Phi_0) \\ &\leq \mathbf{X}(16, 1, 0, 0, 0, 0, -2^{-7}, 1) + 2^{-2}\delta \\ &\leq (\mathbf{X}(26, 1, 1, 1, 1, 0, -2^{-7}, 1) + 2^{-2})\delta \\ &\stackrel{(\bullet)_5}{\leq} \delta \end{aligned}$$

(Note: although in the definition of P we have the difference in ρ 's and in the definition of $d_{\mathcal{E}}$ in ρ^2 's, the estimate above is still ok for (59) since $|\rho^2[\Phi] - \rho^2[\Psi]| \leq |\rho[\Phi] - \rho[\Psi]|$ always holds. To be precise, $|\rho^2[\Phi] - \rho^2[\Psi]| \leq 2\max\{|\rho[\Phi]|, |\rho[\Psi]|\}|\rho[\Phi] - \rho[\Psi]|$). Further, recall Lemma 3.6. We have

$$(60) \quad |\mathbf{I}_S[\Psi, \mathbf{h}, \pi]| \leq \mathbf{X}(13, 1, 0, 0, 0, 0, -2^{-7}, 1) \stackrel{(53)}{\leq} 2^{-6}\delta \mathbf{X}(29, 1, 1, 1, 1, 0, -2^{-7}, 1) \stackrel{(\bullet)_5, (49)}{\leq} 2^{-6}\delta$$

$$(61) \quad |\mathbf{I}_S[\Psi] - \mathbf{I}_S[\Psi']| \leq 2^{-5}\mathbf{X}(24, 2, 0, 0, 0, 0, -2^{-7}, 1)d_{\mathcal{E}}(\Psi, \Psi') \stackrel{(\bullet)_5}{\leq} 2^{-5}d_{\mathcal{E}}(\Psi, \Psi')$$

We now want to apply Banach Fixed Point Theorem to $P(\Psi)$ and, therefore, show that P admits a unique fixed point on $B_{\mathcal{E}}[\frac{1}{2}\delta, \Phi_0]$. For that, we need the following assumptions to be satisfied:

- P is defined on a non-empty complete metric space;

- P is a contraction, i.e. $\exists \text{const. } L < 1$ such that $\forall \Psi, \Psi' \in B_{\mathcal{E}}[\frac{1}{2}\delta, \Phi_0]$: $d_{\mathcal{E}}(P(\Psi), P(\Psi')) \leq L d_{\mathcal{E}}(\Psi, \Psi')$

The first requirement is satisfied by Definitions 3.5 and 3.6. We now show that P is a contraction. Recall $\tau_{0+} = \tau_{1+} = \tau_{2+} < \min\{2, \tau_+\}$. Therefore,

$$(62) \quad \sup_{\tau \in [\tau_{0-}, \tau_{0+}]} |\tau - \tau_{0+}| \leq \sup_{\tau \in [\tau_{0-}, \tau_{0+}]} (|\tau| + |\tau_{0+}|) \leq \sup_{\tau \in [\tau_{0-}, \tau_{0+}]} |\tau| + \sup |\tau_{0+}| \leq 4$$

Recall the definition of $P(\Psi)$ and the right hand sides (RHS) of equations (55)-(59). By (60) and (62) each of the RHS satisfies $\leq 2^{-2}\delta$. Therefore, $P(\Psi) \in B_{\mathcal{E}}[\frac{1}{2}\delta, \Phi_0]$ and is a contraction with $L \leq \frac{1}{2}$.

So, by Banach Fixed Point Theorem it follows that P has a unique fixed point

$$(63) \quad \boxed{\Phi \in B_{\mathcal{E}}[\frac{1}{2}\delta, \Phi_0]}$$

Proof that the fixed point satisfies $(\mathfrak{a}, \mathfrak{b}, c, \mathfrak{d})[\Phi, \mathbf{h}, Z] = 0$. Fixed point is smooth. Observe that $\Phi(\tau_{0+}) = \Phi_1(\tau_{0+}) \stackrel{(52)}{=} \Phi_1(\tau_{1+}) \stackrel{\text{Definition 3.15}}{=} \Phi_$. Further, $c[\Phi, \mathbf{h}, Z](\tau_{0+}) \stackrel{(52)}{=} c[\Phi, \mathbf{h}, Z](\tau_{1+}) \stackrel{\text{Lemma 3.8, b}}{=} 0$.*

Set $\Psi = P(\Psi) = \Phi$ in (55)-(59) and differentiate with respect to τ . Recall Definition 3.11. We have:

for (55) and (56): The result of differentiation both of them can be written in the matrix form (so that we can use Lemma 3.1)

$$\frac{d}{d\tau} \begin{pmatrix} A_a \\ \theta_a \end{pmatrix} = \frac{1}{(A_a)^2} \begin{pmatrix} \frac{1}{\mathbf{h}}(A_a)^2 \tanh \phi_a & \frac{1}{\mathbf{h}}(A_a)^2 \text{sech} \phi_a \\ \phi_a \tanh \phi_a - 1 & \sinh \phi_a + \phi_a \text{sech} \phi_a \end{pmatrix} \begin{pmatrix} \mathfrak{a}_a[\Phi, \mathbf{h}, B_a] - \mathfrak{a}_a[\Phi, \mathbf{h}, Z] \\ -(\sigma_*)_a \mathfrak{b}_a[\Phi, \mathbf{h}, B_a] + (\sigma_*)_a \mathfrak{b}_a[\Phi, \mathbf{h}, Z] \end{pmatrix}$$

Comparing it with the equation in Lemma 3.1, we conclude that $\boxed{\mathfrak{a}_a[\Phi, \mathbf{h}, Z] = \mathfrak{b}_a[\Phi, \mathbf{h}, Z] = 0}$ must hold.

for (57): The result of differentiation is

$$(64) \quad \frac{d}{d\tau} \alpha_{p,a}[\Phi] = \frac{1}{\mathbf{h}} \mathfrak{a}_p[\Phi, \mathbf{h}, Z, B_a] + \frac{1}{\mathbf{h}} \mathfrak{a}_a[\Phi, \mathbf{h}, Z, B_a]$$

Recall equation (24), which implies (no energy density terms appear because B_a has $m = 0$)

$$(65) \quad \mathfrak{a}_p[\Phi, \mathbf{h}, B_a] + \mathfrak{a}_a[\Phi, \mathbf{h}, B_a] = -\mathbf{h} \frac{d}{d\tau} \alpha_{p,a}[\Phi]$$

Using the earlier obtained result $\mathfrak{a}_a[\Phi, \mathbf{h}, Z] = 0$ we calculate

$$\mathfrak{a}_p[\Phi, \mathbf{h}, B_a] + \mathfrak{a}_a[\Phi, \mathbf{h}, B_a] \stackrel{(65)}{=} -\mathbf{h} \frac{d}{d\tau} \alpha_{p,a}[\Phi] \stackrel{(64)}{=} -(\mathfrak{a}_p[\Phi, \mathbf{h}, Z] - \mathfrak{a}_p[\Phi, \mathbf{h}, B_a] + \underbrace{\mathfrak{a}_a[\Phi, \mathbf{h}, Z] - \mathfrak{a}_a[\Phi, \mathbf{h}, B_a]}_{=0})$$

which implies $\boxed{\mathfrak{a}_p[\Phi, \mathbf{h}, Z] = 0}$.

for (58): The result of differentiation and (64) give $\frac{d}{d\tau} \xi_{p,a}[\Phi, \mathbf{h}] = \alpha_{p,a}[\Phi]$. The explicit calculation gives

$$\begin{aligned} \alpha_p[\Phi] + \alpha_a[\Phi] &= \frac{d}{d\tau} (\xi_p[\Phi] + \xi_a[\Phi]) \\ &\stackrel{\text{Definition 3.3}}{=} \frac{d}{d\tau} (\mathbf{h} \log |\frac{1}{2} \beta_p[\Phi]|) + \frac{d}{d\tau} (\mathbf{h} \log |\frac{1}{2} \beta_a[\Phi]|) \\ &= \mathbf{h} \left(\frac{1}{|\beta_p|} \frac{d}{d\tau} |\beta_p| + \frac{1}{|\beta_a|} \frac{d}{d\tau} |\beta_a| \right) \end{aligned}$$

which is equivalent to

$$-\beta_a[\Phi] \left(-\mathbf{h} \frac{d}{d\tau} \beta_p[\Phi] + \alpha_p[\Phi] \beta_p[\Phi] \right) - \beta_p[\Phi] \underbrace{\left(-\mathbf{h} \frac{d}{d\tau} \beta_a[\Phi] + \alpha_a[\Phi] \beta_a[\Phi] \right)}_{=\mathfrak{b}_a[\Phi, \mathbf{h}, Z]=0} = 0$$

Therefore, $\boxed{\mathfrak{b}_p[\Phi, \mathbf{h}, Z] = 0}$.

for (59): The result of differentiation gives $\frac{d}{d\tau} \rho[\Phi] = 0$, which by Definition 3.3 is equivalent to $\boxed{\mathfrak{d}[\Phi, \mathbf{h}, Z] = 0}$.

Now, Proposition 3.2 and $\mathfrak{d}[\Phi, \mathbf{h}, Z] = 0$ imply that $\boxed{c[\Phi, \mathbf{h}, Z] = 0}$ on $[\tau_{0-}, \tau_{0+}]$.

Estimates on Φ . We have $P(\Phi) = \Phi$ by the fixed point equation. Further,

$$\begin{aligned} d_{\mathcal{E}}(\Phi_0, \Phi) &\leq d_{\mathcal{E}}(\Phi_0, \Phi_1) + d_{\mathcal{E}}(\Phi_1, \Phi) \\ &= d_{\mathcal{E}}(\Phi_0, \Phi_1) + d_{\mathcal{E}}(\Phi_1, P(\Phi)) \\ &\stackrel{\text{Preliminaries 2, (60)}}{\leq} \mathbf{X}(19, 0, 0, 1, 0, 0, -2^{-1}, 0) + \mathbf{X}(17, 1, 0, 0, 0, 0, -2^{-7}, 1) \\ &\leq \mathbf{X}(20, 1, 0, 1, 0, 0, -2^{-7}, 1) \end{aligned}$$

The result above motivates to require $(\ell_1, \dots, \ell_8) \geq (20, 1, 0, 1, 0, 0, -2^{-7}, 1)$ $(\bullet)_6$ so that we can have (recall equation (49)) $d_{\mathcal{E}}(\Phi_0, \Phi) \leq \mathbf{K}$. We now want to make an estimate on the metric space $(\mathcal{D}(\sigma_*), \mathcal{d}_{\mathcal{D}(\sigma_*)}, \mathbf{h})$. That is, we want an estimate for $\mathcal{d}_{\mathcal{D}}(\Phi_0(\tau), \Phi(\tau))$ for τ in a sub-interval of $[\tau_{0-}, \tau_{0+}]$. The desired bound for $\mathcal{d}_{\mathcal{D}}(\Phi_0(\tau), \Phi(\tau))$ comes from Lemma 3.2 a). Define $\mathcal{J} := [\tau_{0-}, \frac{1}{2}\tau_{1-}] \subset [\tau_{0-}, \tau_{0+}]$. Morally, \mathcal{J} represents the interval that allowed

us "extra space" for the fixed point construction. The latter was technically carried out on $[\tau_{0-}, \tau_{0+}]$, but the idea was "from crossing to crossing". To the right, we had exactly $\tau_{0+} = \tau_{1+}$, but on the left $\tau_{0-} < \tau_{1-}$. That difference is represented by \mathcal{J} . Setting $C = 2$ and $D = 12\max\{1, \frac{1}{q}\}$ the assumptions of Lemma 3.2 are satisfied. Observe that $C^2D \leq \mathbf{X}(6, 0, 0, 0, 1, 0, 0, 0)$ and we estimate

$$(66) \quad \mathcal{A}_{\mathcal{D}}(\Phi_0(\tau), \Phi(\tau)) \leq 2^3 C^2 D d_{\mathcal{E}}(\Phi_0, \Phi) \leq \mathbf{X}(29, 1, 0, 1, 1, 0, -2^{-7}, 1) =: \mathbf{M}$$

Construction of τ_{2-} . Recall that $a' = c$ if $q < 1$ and $a' = b$ if $q > 1$. Recall Lemma 3.9. For all $\tau \in \mathcal{J}$ we have

$$(67) \quad (\xi_a[\Phi_0, \mathbf{h}] - \xi_{a'}[\Phi_0, \mathbf{h}])F = \tau - \tau_{1-} - \underbrace{2\mathbf{h}\log(1 + e^{2\tau/\mathbf{h}})F}_{=: T_1}$$

By adding and subtracting $\xi_a[\Phi, \mathbf{h}]$ and $\xi_{a'}[\Phi, \mathbf{h}]$ to the LHS of (67) we get

$$(68) \quad (\xi_a[\Phi, \mathbf{h}] - \xi_{a'}[\Phi, \mathbf{h}])F = \tau - \tau_{1-} - T_2$$

with $T_2 := T_1 - (\xi_a[\Phi, \mathbf{h}] - \xi_a[\Phi_0, \mathbf{h}])F + (\xi_{a'}[\Phi, \mathbf{h}] - \xi_{a'}[\Phi_0, \mathbf{h}])F$. Both T_1 and T_2 are functions of $\tau \in \mathcal{J}$. For all such τ we now want to make an estimate for the RHS of (68). That is, how $|\tau - \tau_{1-}|$ compares to $|T_2|$. We have

$$0 < T_1 \stackrel{Def.}{=} 2\mathbf{h}\log(1 + e^{2\tau/\mathbf{h}})F \stackrel{\log(1+x) \leq x}{\leq} 2\mathbf{h}e^{2\tau/\mathbf{h}}F \stackrel{\tau \leq \frac{1}{2}\tau_{1-}}{\leq} 2\mathbf{h}e^{\tau_{1-}/\mathbf{h}}F \leq \mathbf{M}F$$

Therefore,

$$\begin{aligned} |T_2| &= |T_1 - (\xi_a[\Phi, \mathbf{h}] - \xi_a[\Phi_0, \mathbf{h}])F + (\xi_{a'}[\Phi, \mathbf{h}] - \xi_{a'}[\Phi_0, \mathbf{h}])F| \\ &\leq |T_1| + |\xi_a[\Phi, \mathbf{h}] - \xi_a[\Phi_0, \mathbf{h}]|F + |\xi_{a'}[\Phi, \mathbf{h}] - \xi_{a'}[\Phi_0, \mathbf{h}]|F \\ &\leq \mathbf{M}F + 2 \mathcal{A}_{\mathcal{D}}(\Phi, \Phi_0)F \\ &\leq 3\mathbf{M}F \\ &\stackrel{F \leq \frac{1}{2} \text{ by Def.}}{\leq} \frac{3}{2}\mathbf{M} \end{aligned}$$

Set

$$\text{dist}_{\mathbb{R}}(\tau_{1-}, \mathbb{R} \setminus \mathcal{J}) = \min\{\frac{1}{2}|\tau_{1-}|, \epsilon_{-}\} \stackrel{(51)}{\geq} 2^{-5}\tau_{*} \geq \mathbf{X}(-6, 0, 0, -1, -1, 0, 0, 0)$$

Then

$$\begin{aligned}
|T_2| &\leq \frac{3}{2} \mathbf{X}(29, 1, 0, 1, 1, 0, -2^{-7}, 1) \\
&= \frac{3}{2} \mathbf{X}(-6, 0, 0, -1, -1, 0, 0, 0) \mathbf{X}(35, 1, 0, 2, 2, 0, -2^{-7}, 1) \\
&\leq \mathbf{X}(37, 1, 0, 2, 2, 0, -2^{-7}, 1) \text{dist}_{\mathbb{R}}(\tau_{1-}, \mathbb{R} \setminus \mathcal{J})
\end{aligned}$$

Require $(\ell_1, \dots, \ell_8) \geq (37, 1, 0, 2, 2, 0, -2^{-7}, 1) (\bullet)_7$. Therefore, $|T_2| \leq \text{dist}_{\mathbb{R}}(\tau_{1-}, \mathbb{R} \setminus \mathcal{J})$.

Set

$$(69) \quad \tau_{2-} = \sup\{\tau \in \mathcal{J} | \xi_a[\Phi, \mathbf{h}](\tau) \leq \xi_{a'}[\Phi, \mathbf{h}](\tau)\}$$

The rest stays the same. The condition $(\ell_1, \dots, \ell_8) \geq (31, 1, 0, 1, 1, 0, -2^{-7}, 1) (\bullet)_8$ implies $|\tau_{2-} - \tau_{1-}| \leq \frac{3}{2} \mathbf{M}^{(\bullet)_8} \leq \mathbf{K}$.

Estimates on Φ_0 . Here, we want to estimate how big were the terms that we neglected for α and ξ . For example, we assumed that $\xi_a(\tau) \approx \tau$, where the " \approx " meant neglecting the $\mathcal{O}(\exp(-2|\tau|))$ term for the limiting cases $\tau \gg \mathbf{h}$ and $\tau \ll \mathbf{h}$. We now make it quantitative. For all $\tau \in \mathcal{J}$ we have

$$\begin{aligned}
|\alpha_a[\Phi_0](\tau) - 1| &= \left| \tanh \frac{1}{\mathbf{h}} |\tau| - 1 \right| \leq 2 \exp \left(-\frac{2}{\mathbf{h}} |\tau| \right) \\
|\xi_a[\Phi_0, \mathbf{h}](\tau) - \tau| &= \left| \mathbf{h} \log(2 \cosh \frac{1}{\mathbf{h}} |\tau|) - |\tau| \right| \leq \mathbf{h} \exp \left(-\frac{2}{\mathbf{h}} |\tau| \right) \\
\exp \left(-\frac{2}{\mathbf{h}} |\tau| \right) &\leq \exp \left(-\frac{1}{\mathbf{h}} |\tau_{1-}| \right) \leq \exp \left(-\frac{1}{\mathbf{h}} 2^{-2} \tau_* \right) \leq 2^{-29} \mathbf{M}
\end{aligned}$$

Construction of λ . Set

$$(70) \quad \lambda = \alpha_{a'}[\Phi](\tau_{2-})$$

Recall Lemma 3.9. Then,

$$\begin{aligned}
|\lambda - \lambda_L| &= |\alpha_{a'}[\Phi](\tau_{2-}) - (1 - \alpha_{a,a'}[\Phi_0](\tau_{1-}))| \\
&\leq |\alpha_{a'}[\Phi](\tau_{2-}) - \alpha_{a'}[\Phi_0](\tau_{2-})| + |\alpha_{a'}[\Phi_0](\tau_{2-}) - \alpha_{a'}[\Phi_0](\tau_{1-})| + |\alpha_{a'}[\Phi_0](\tau_{1-}) + (1 - \alpha_{a,a'}[\Phi_0](\tau_{1-}))| \\
&\leq |\alpha_{a'}[\Phi](\tau_{2-}) - \alpha_{a'}[\Phi_0](\tau_{2-})| + (|\alpha_a[\Phi_0](\tau_{2-}) - 1| + |\alpha_a[\Phi_0](\tau_{1-}) - 1|) + |1 - \alpha_a[\Phi_0](\tau_{1-})| \\
&\leq d_{\mathcal{E}}(\Phi, \Phi_0) + 6 \cdot 2^{-29} \mathbf{M} \\
&\stackrel{(66)}{\leq} 2\mathbf{M}
\end{aligned}$$

Require $(\ell_1, \dots, \ell_8) \geq (32, 1, 0, 2, 1, 0, -2^{-7}, 1) (\bullet)_9$. If it holds, then $\lambda \geq \lambda_L - (1+w)^{-1} \geq 1$.

Construction of z' . Recall equation (46). Set

$$(71) \quad z' = -\frac{\rho^2[\Phi](\tau_{2-})}{\alpha_{a'}[\Phi](\tau_{2-})}$$

Then,

$$\begin{aligned}
|z' - z_L| &= \left| -\frac{\rho^2[\Phi]}{\alpha_{a'}[\Phi](\tau_{2-})} - \frac{z}{1 - \alpha_{a,a'}[\Phi_0](\tau_{1-})} \right| \\
&= \left| \frac{\rho^2[\Phi](\tau_{2-})}{\lambda} - \frac{\rho^2[\Phi_0](\tau_{1-})}{\lambda_L} \right| \\
&= \left| \frac{\rho^2[\Phi](\tau_{2-})\lambda_L - \rho^2[\Phi_0](\tau_{1-})\lambda}{\lambda\lambda_L} \right| \\
&\leq \frac{1}{|\lambda\lambda_L|} (\rho^2[\Phi](\tau_{2-})|\lambda - \lambda_L| + |\lambda| |\rho^2[\Phi](\tau_{2-}) - \rho^2[\Phi_0](\tau_{1-})|) \\
&\leq \frac{1}{|\lambda\lambda_L|} (\rho^2[\Phi](\tau_{2-})|\lambda - \lambda_L| + |\lambda| |\rho^2[\Phi](\tau_{2-}) - \rho^2[\Phi_0](\tau_{2-})|) \\
&\leq \frac{1}{|\lambda\lambda_L|} 2z\mathbf{M} + \frac{1}{|\lambda_L|} \mathbf{M} \\
&\leq 2z\mathbf{M} + \mathbf{M} \\
&= \mathbf{X}(30, 1, 0, 1, 1, 0, -2^{-7}, 2) + \mathbf{X}(29, 1, 0, 1, 1, 0, -2^{-7}, 1) \\
&\leq \mathbf{X}(31, 1, 0, 1, 1, 0, -2^{-7}, 2)
\end{aligned}$$

where $\lambda \geq \lambda_L - \frac{1}{1+w} \geq 1$ was used, which also implies $|\lambda\lambda_L| \neq 0$, $\frac{1}{|\lambda_L|} \leq \frac{1}{1+\frac{1}{1+w}} < 1$,

and $|\lambda\lambda_L| \geq 1 + \frac{1}{1+w}$.

Require $(\ell_1, \dots, \ell_8) \geq (31, 1, 0, 1, 1, 0, -2^{-7}, 2) (\bullet)_{10}$. Then, $|z' - z_L| \leq 1$.

Construction of w' . (stays the same as in [2]) Require $(\ell_1, \dots, \ell_8) \geq (32, 1, 0, 2, 1, 0, -2^{-7}, 1)$ $(\bullet)_{11}$. Set

$$(72) \quad w' = \frac{\alpha_a[\Phi](\tau_{2-})}{-\alpha_{a,a'}[\Phi](\tau_{2-})} > 0$$

To check that w' is well-defined, i.e. denominator is non-zero and $w' > 0$, recall *Estimates on Φ_0* and (66). We have

$$\begin{aligned} |\alpha_{a,a'}[\Phi](\tau) - \alpha_{a,a'}[\Phi_0](\tau)| &\leq 2\mathbf{M} \\ |\alpha_a[\Phi](\tau) - 1| &= |\alpha_a[\Phi](\tau) - \alpha_a[\Phi_0](\tau) + \alpha_a[\Phi_0](\tau) - 1| \leq |\alpha_a[\Phi](\tau) - \alpha_a[\Phi_0](\tau)| + |\alpha_a[\Phi_0](\tau) - 1| \end{aligned}$$

Further, $4\mathbf{M} \leq \mathbf{X}(-1, 0, 0, -1, 0, 0, 0, 0) \leq \frac{1}{1+w} \stackrel{(32),(33)}{\leq} |\alpha_{a,a'}[\Phi_0](\tau)|$. Recall $(\bullet)_9$, then $4\mathbf{M} < 1$. We estimate

$$\begin{aligned} |\alpha_{a,a'}[\Phi](\tau) - \alpha_{a,a'}[\Phi_0](\tau)| &\leq \frac{1}{2} |\alpha_{a,a'}[\Phi_0](\tau)| \\ |\alpha_a[\Phi](\tau) - 1| &\leq \frac{1}{2} \end{aligned}$$

The last inequality implies $\frac{1}{2} \leq \alpha_a[\Phi](\tau) \leq \frac{3}{2}$ for all $\tau \in \mathcal{J}$, in particular, $\alpha_a[\Phi](\tau_{2-}) > 0$. From the second to last inequality we get $\alpha_{a,a'}[\Phi](\tau_{2-}) \leq \frac{1}{2} \alpha_{a,a'}[\Phi_0](\tau_{2-}) < 0$. The last inequality follows from (32),(33). Therefore, w' is well-defined.

Recall Lemma 3.9. We estimate

$$\begin{aligned} |w' - w_L| &= \left| \frac{\alpha_a[\Phi](\tau_{2-})}{-\alpha_{a,a'}[\Phi](\tau_{2-})} - \left(-\frac{1}{\alpha_{a,a'}[\Phi_0](\tau_{2-})} \right) \right| \\ &\leq \left| \frac{\alpha_a[\Phi](\tau_{2-})}{-\alpha_{a,a'}[\Phi](\tau_{2-})} - \frac{\alpha_a[\Phi_0](\tau_{2-})}{-\alpha_{a,a'}[\Phi](\tau_{2-})} \right| \\ &\quad + \left| \frac{\alpha_a[\Phi_0](\tau_{2-})}{-\alpha_{a,a'}[\Phi](\tau_{2-})} - \frac{\alpha_a[\Phi_0](\tau_{2-})}{-\alpha_{a,a'}[\Phi_0](\tau_{2-})} \right| + \left| \frac{\alpha_a[\Phi_0](\tau_{2-})}{-\alpha_{a,a'}[\Phi_0](\tau_{2-})} - \frac{1}{\alpha_{a,a'}[\Phi_0](\tau_{2-})} \right| \\ &\leq \frac{1}{|\alpha_{a,a'}[\Phi](\tau_{2-})|} |\alpha_a[\Phi](\tau_{2-}) - \alpha_a[\Phi_0](\tau_{2-})| + \left| \frac{\alpha_a[\Phi_0](\tau_{2-})}{-\alpha_{a,a'}[\Phi](\tau_{2-})} - \frac{\alpha_a[\Phi_0](\tau_{2-})}{-\alpha_{a,a'}[\Phi_0](\tau_{2-})} \right| + \\ &\quad + \left| \frac{\alpha_a[\Phi_0](\tau_{2-})}{-\alpha_{a,a'}[\Phi_0](\tau_{2-})} - \frac{1}{\alpha_{a,a'}[\Phi_0](\tau_{2-})} \right| \\ &\leq 2w_L \mathbf{M} + 4w_L^2 \mathbf{M} + w_L \mathbf{M} \\ &\leq 2^3(1+w)^2 \mathbf{M} \\ &\leq \frac{1}{2} \mathbf{X}(6, 0, 0, 2, 0, 0, 0, 0) \mathbf{M} \end{aligned}$$

Require $(\ell_1, \dots, \ell_8) \geq (35, 1, 0, 3, 1, 0, -2^{-7}, 1)$ $(\bullet)_{12}$. Therefore, $|w' - w_L| \leq \frac{1}{2}\mathbf{K} \leq \frac{1}{2}$.

Construction of \mathbf{h}' . Stays the same as in [2]. Inequalities $(\bullet)_{13}, (\bullet)_{14}, (\bullet)_{15}$ will be added. Main result:

$$(73) \quad \mathbf{h}' = \frac{\mathbf{h}}{\mu} > 0$$

where $\mu = \frac{1+2w'}{1+w'}(-\xi_a[\Phi, \mathbf{h}](\tau_{2-}) + \mathbf{h} \log \lambda) - \mathbf{h} \log 2$. Further, $|\mathbf{h}' - \mathbf{h}_L| \leq \frac{1}{2}$.

Construction of q' . Stays the same as in [2]. Inequality $(\bullet)_{16}$ will be added. Main result:

$$(74) \quad q' = \frac{1}{1+w'} \left(\mathbf{h}' \log \lambda - \frac{\mathbf{h}'}{\mathbf{h}} \xi_{c'}[\Phi, \mathbf{h}](\tau_{2-}) - \frac{w'(1+w')}{1+2w'} - \frac{1+3w'+(w')^2}{1+2w'} \mathbf{h}' \log 2 \right)$$

The maximum of $\alpha_{b,c}, \alpha_{c,a}, \alpha_{a,b}$. Recall *Estimates on Φ* . We had $d_{\mathcal{E}}(\Phi, \Phi_0) \leq \mathbf{X}(20, 1, 0, 1, 0, 0, -2^{-7}, 1)$. Recall Definition 3.6. We have

$$\begin{aligned} \alpha_{a,p}[\Phi] &\leq \alpha_{a,p}[\Phi_0] + \alpha_a[\Phi] + \alpha_p[\Phi] - \alpha_a[\Phi_0] - \alpha_p[\Phi_0] \\ &\stackrel{(32),(33)}{\leq} -\frac{1}{1+w} + d_{\mathcal{D}}(\Phi_0, \Phi) \\ &\leq -\frac{1}{1+w} + \mathbf{X}(20, 1, 0, 1, 0, 0, -2^{-7}, 1) \end{aligned}$$

and

$$\begin{aligned} \alpha_{b,c}[\Phi] &\leq \alpha_{a,b}[\Phi] + \alpha_{a,c}[\Phi] - 2\alpha_a[\Phi] \\ &\stackrel{\text{Definition 3.3}}{\leq} \alpha_{a,b}[\Phi] + \alpha_{a,c}[\Phi] + 2A_a[\Phi] \\ &\leq 2^2 d_{\mathcal{D}}(\Phi_0, \Phi) + \alpha_{a,b}[\Phi_0] + \alpha_{a,c}[\Phi_0] + 2A_a[\Phi_0] \\ &\leq 2^2 \mathbf{X}(20, 1, 0, 1, 0, 0, -2^{-7}, 1) + \alpha_{a,b}[\Phi_0] + \alpha_{a,c}[\Phi_0] + 2A_a[\Phi_0] \end{aligned}$$

Now, require $(\ell_1, \dots, \ell_8) \geq (24, 1, 2, 2, 0, 0, -2^{-7}, 1)$ $(\bullet)_{17}$. Then $\mathbf{X}(20, 1, 0, 1, 0, 0, -2^{-7}, 1) \leq 2^{-3}(1+w)^{-1} \min\{w^2, 1\}$ and $d_{\mathcal{D}}(\Phi_0(\tau), \Phi(\tau)) \leq 2^{-3}(1+w)^{-1} \min\{w^2, 1\}$ for all $\tau \in [\tau_{0-}, \tau_{0+}]$. This implies that

$$\alpha_{a,p}[\Phi] \leq -\frac{1}{1+w} + 2^{-3} \frac{1}{1+w} \min\{w^2, 1\} \leq -2^{-1}(1+w)^{-1}$$

and

$$\alpha_{b,c}[\Phi] \stackrel{(30),(32),(33)}{\leq} -2^{-1}(1+w)^{-1}w^2$$

for all $\tau \in [\tau_{0-}, \tau_{0+}]$ and $p = \{b, c\}$.

Definitions of maps Π, Λ and τ_{2-} . Set $(\ell_1, \dots, \ell_8) = (40, 2, 2, 3, 2, 1, -2^{-7}, 2)$ and $\ell_9 = 7$. With this choice, all inequalities (\bullet) hold. The constant \mathbf{K} defined by (49) coincides with $\mathbf{K}(\mathbf{f})$ defined in (48). Vector $\mathbf{f} = (\mathbf{h}, w, q, z) \in (0, \infty)^4$ satisfies (49) if and only if $\mathbf{f} \in \mathcal{F}$. Therefore, we can set

$$\begin{aligned} \Pi[\pi, \sigma_*] : \mathcal{F} &\rightarrow (0, \infty)^2 \times \mathbb{R} \times (0, \infty) & \mathbf{f} &\mapsto \text{right hand side of (73), (72), (74), (71)} \\ \Lambda[\pi, \sigma_*] : \mathcal{F} &\rightarrow [1, \infty) & \mathbf{f} &\mapsto \text{right hand side of (70)} \\ \tau_{2-}[\pi, \sigma_*] : \mathcal{F} &\rightarrow (\infty, 0) & \mathbf{f} &\mapsto \text{right hand side of (69)} \end{aligned}$$

Equation $\Phi(\tau_{2-}) = \lambda \Phi_*(\pi', \mathbf{f}', \sigma_*)$ with $\mathbf{f}' = (\mathbf{h}', w', q', z')$ is equivalent to (recall $b' = a$)

$$\begin{aligned} \alpha_{a'}[\Phi](\tau_{2-}) &= -\lambda \\ \alpha_a[\Phi](\tau_{2-}) &= \lambda \frac{w'}{1+w'} \\ \alpha_{c'}[\Phi](\tau_{2-}) &= \lambda(-w' - \mu' - 4(1+w')\gamma^2) \\ \frac{1}{\mathbf{h}}\xi_{a'}[\Phi, \mathbf{h}](\tau_{2-}) &= \log \lambda + \frac{1}{\mathbf{h}'} \left(-\frac{1+w'}{1+2w'}(1 + \mathbf{h}' \log 2) \right) \\ \frac{1}{\mathbf{h}}\xi_a[\Phi, \mathbf{h}](\tau_{2-}) &= \log \lambda + \frac{1}{\mathbf{h}'} \left(-\frac{1+w'}{1+2w'}(1 + \mathbf{h}' \log 2) \right) \\ \frac{1}{\mathbf{h}}\xi_{c'}[\Phi, \mathbf{h}](\tau_{2-}) &= \log \lambda + \frac{1}{\mathbf{h}'} \left(-(1+w')q' - \frac{w'(1+w')}{1+2w'} - \frac{1+3w'+(w')^2}{1+2w'} \mathbf{h}' \log 2 \right) \\ \rho^2[\Phi](\tau_{2-}) &= \lambda z' \end{aligned}$$

with $\mu' = (1+w')(\beta_1^2 + \beta_2^2 + \beta_3^2 - 2\beta_2\beta_3 - 2\beta_3\beta_1 - 2\beta_1\beta_2)|_{\beta=\beta[\Phi_*(\pi', \mathbf{f}', \sigma_*)]}$ and $\gamma^2 = \rho^2|\beta_1\beta_2\beta_3|^{1/3}|_{\beta=\beta[\Phi_*(\pi', \mathbf{f}', \sigma_*)]}$

Continuity of maps Π, Λ and τ_{2-} . Fix $\mathbf{f}^\Psi = (\mathbf{h}^\Psi, w^\Psi, q^\Psi, z^\Psi) \in \mathcal{F}$. Let $r > 0$ and let $\mathbf{f}^\Upsilon = (\mathbf{h}^\Upsilon, w^\Upsilon, q^\Upsilon, z^\Upsilon) \in \mathcal{F}$ with $\|\mathbf{f}^\Psi - \mathbf{f}^\Upsilon\|_{\mathbb{R}} \leq r$. Introduce notation $\mathbf{f}^B \in \mathcal{F}$ with $B = \Psi, \Upsilon$. Following this convention, the contraction mapping fixed points are denoted $\Phi^B \in \mathcal{E}^B$. We also write $\Phi^\Psi = \Psi$ and $\Phi^\Upsilon = \Upsilon$.

Suppose $r \leq \frac{1}{2}|q^\Psi - 1|$. Then $0 \neq \text{sgn}(q^\Psi - 1) = \text{sgn}(q^\Upsilon - 1)$.

Now, if we want to compare the solutions Ψ and Υ , we have to apply the symmetry transformation from Proposition 3.1 to make sure they solve the same equation. That is, we introduce a new field of the form " $\Xi = \Upsilon \circ \chi$ ". Recall that in the Proposition 3.1 we had the general form $\chi(\tau) = p\tau + q$. Here, introduce $\chi : \mathbb{R} \rightarrow \mathbb{R}$ by

$$(75) \quad \chi(\tau) = \frac{\mathbf{h}^\Upsilon}{\mathbf{h}^\Psi}(\tau - \tau_{0+}^\Psi) + \tau_{0+}^\Upsilon$$

The motivation for such form of χ is as follows. The fixed point construction was carried out on the interval $[\tau_{0-}, \tau_{0+}]$. Here, introduce $\mathcal{I}^B = [\tau_{0-}^B, \tau_{0+}^B]$ with $B = \Psi, \Upsilon$, and $\mathcal{I}^\Xi = [\chi^{-1}(\tau_{0-}^\Upsilon), \tau_{0+}^\Psi]$. With these definitions, we have $\chi(\mathcal{I}^\Xi) = \mathcal{I}^\Upsilon$, and by Proposition 3.1 the field $\Xi = \Upsilon \circ (\chi|_{\mathcal{I}^\Xi})$ satisfies $(\mathbf{a}, \mathbf{b}, c, \mathbf{d})[\Xi, \mathbf{h}^\Psi, Z] = 0$ on \mathcal{I}^Ξ .

Define $\mathcal{I} = \mathcal{I}^\Psi \cap \mathcal{I}^\Xi$. Recall the definition of \mathcal{J} and set $\mathcal{J}^B = [\tau_{0-}^B, \frac{1}{2}\tau_{1-}^B] \subset \mathcal{I}^B$. Further, $|\tau_{2-}^B - \tau_{1-}^B| \leq \frac{1}{2}\text{dist}_{\mathbb{R}}(\tau_{1-}^B, \mathbb{R} \setminus \mathcal{J}^B)$.

Define $\mathcal{J} = \mathcal{J}^\Psi \cap \mathcal{J}^\Xi \subset \mathcal{I}$, with $\mathcal{J}^\Xi = \chi^{-1}(\mathcal{J}^\Upsilon)$. For $r > 0$ sufficiently small, we want to show

- $\tau_{2-}^\Psi \in \mathcal{J}$
- $\chi^{-1}(\tau_{2-}^\Upsilon) \in \mathcal{J}$

To prove the first one, we need to show that $\tau_{2-}^\Psi \in \mathcal{J}^\Psi$ and $\tau_{2-}^\Psi \in \mathcal{J}^\Xi$. Clearly, $\tau_{2-}^\Psi \in \mathcal{J}^\Psi$. The inclusion $\tau_{2-}^\Psi \in \mathcal{J}^\Xi$ is equivalent to $\chi(\tau_{2-}^\Psi) \in \mathcal{J}^\Upsilon$ by definition of $\chi(\tau)$. So, the goal is to show that $|\chi(\tau_{2-}^\Psi) - \tau_{1-}^\Upsilon| \leq \frac{1}{2}\text{dist}_{\mathbb{R}}(\tau_{1-}^\Upsilon, \mathbb{R} \setminus \mathcal{J}^\Upsilon)$. We have

$$\begin{aligned} |\chi(\tau_{2-}^\Psi) - \tau_{1-}^\Upsilon| &\leq |\chi(\tau_{2-}^\Psi) - \chi(\tau_{1-}^\Psi)| + |\chi(\tau_{1-}^\Psi) - \tau_{1-}^\Upsilon| \\ &\leq \frac{\mathbf{h}^\Upsilon}{\mathbf{h}^\Psi} \frac{1}{2} \text{dist}_{\mathbb{R}}(\tau_{1-}^\Psi, \mathbb{R} \setminus \mathcal{J}^\Psi) + |\chi(\tau_{1-}^\Psi) - \tau_{1-}^\Upsilon| \\ &\stackrel{\text{for } \mathbf{f}^\Upsilon = \mathbf{f}^\Psi}{=} \frac{1}{2} \text{dist}_{\mathbb{R}}(\tau_{1-}^\Upsilon, \mathbb{R} \setminus \mathcal{J}^\Upsilon) \end{aligned}$$

Note that the RHS of the second inequality is a continuous function of \mathbf{f}^Υ .

Introduce abbreviations $\mathcal{D} = \mathcal{D}^\Psi = \mathcal{D}^\Upsilon = \mathcal{D}(\sigma_*)$, $\mathcal{E} = \mathcal{E}(\sigma_*; \mathcal{I})$, $\Phi_0 = \Phi_0(\pi, \mathbf{f}^\Psi, \sigma_*)|_{\mathcal{I}} = \Phi_0^\Psi|_{\mathcal{I}}$, $d_{\mathcal{X}} = d_{\mathcal{X}, (\pi, \mathbf{h}^\Psi)}$ for $\mathcal{X} = \mathcal{E}, \mathcal{D}$, $d_{\mathcal{X}^B} = d_{\mathcal{X}^B, (\pi, \mathbf{h}^B)}$ for $B = \Psi, \Upsilon$.

Recall (63). We have $d_{\mathcal{E}^B}(B, \Phi_0^B) \leq \frac{1}{2}\delta^B$. If $r > 0$ is sufficiently small then we have

$$d_{\mathcal{E}}(\Psi|_{\mathcal{I}}, \Phi_0) \leq d_{\mathcal{E}^\Psi}(\Psi, \Phi_0^\Psi) \leq \frac{1}{2}\delta^\Psi$$

and

$$\begin{aligned}
d_{\mathcal{E}}(\Xi|_{\mathcal{I}}, \Phi_0) &\stackrel{\Xi=\Upsilon \circ \chi|_{\mathcal{I}}}{\leq} d_{\mathcal{E}}(\Upsilon \circ \chi|_{\mathcal{I}}, \Phi_0^{\Upsilon} \circ \chi|_{\mathcal{I}}) + d_{\mathcal{E}}(\Phi_0^{\Upsilon} \circ \chi|_{\mathcal{I}}, \Phi_0) \\
&\stackrel{(75)}{\leq} \max\{1, \frac{\mathbf{h}^{\Psi}}{\mathbf{h}^{\Upsilon}}\} d_{\mathcal{E}^{\Upsilon}}(\Upsilon, \Phi_0^{\Upsilon}) + d_{\mathcal{E}}(\Phi_0^{\Upsilon} \circ \chi|_{\mathcal{I}}, \Phi_0) \\
&\leq \max\{1, \frac{\mathbf{h}^{\Psi}}{\mathbf{h}^{\Upsilon}}\} \frac{1}{2} \delta^{\Upsilon} + d_{\mathcal{E}}(\Phi_0^{\Upsilon} \circ \chi|_{\mathcal{I}}, \Phi_0)
\end{aligned}$$

Note that the RHS of the last inequality is a continuous function of \mathbf{f}^{Υ} and is equal to $\frac{1}{2}\delta^{\Psi}$ when $\mathbf{f}^{\Upsilon} = \mathbf{f}^{\Psi}$. Therefore, we have $d_{\mathcal{E}}(\Psi|_{\mathcal{I}}, \Phi_0) \leq \delta^{\Psi}$ and $d_{\mathcal{E}}(\Xi|_{\mathcal{I}}, \Phi_0) \leq \delta^{\Psi}$.

Now, recall equations (55)-(59). Recall Definition 3.15. Both $X = \Psi|_{\mathcal{I}}$ and $X = \Xi|_{\mathcal{I}}$ satisfy $(\mathbf{a}, \mathbf{b}, c, \mathbf{d})[X, \mathbf{h}^{\Psi}, Z] = 0$ on \mathcal{I} . For all $p \in \{b, c\}$ and $\tau \in \mathcal{I}$, we have

$$\begin{aligned}
A_a[X](\tau) &= A_a[X](\tau_{0+}^{\Psi}) + \int_{\tau_{0+}^{\Psi}}^{\tau} d\tau' \mathbf{I}_1[\Psi, \mathbf{h}, \pi](\tau') \\
\theta_a[X](\tau) &= \theta_a[X, \mathbf{h}^{\Psi}](\tau_{0+}^{\Psi}) + \int_{\tau_{0+}^{\Psi}}^{\tau} d\tau' \mathbf{I}_2[\Psi, \mathbf{h}, \pi](\tau') \\
\alpha_{p,a}[X](\tau) &= \alpha_{p,a}[X](\tau_{0+}^{\Psi}) + \int_{\tau_{0+}^{\Psi}}^{\tau} d\tau' \mathbf{I}_{(3,p)}[\Psi, \mathbf{h}, \pi](\tau') \\
\xi_{p,a}[X](\tau) &= \xi_{p,a}[X, \mathbf{h}^{\Psi}](\tau_{0+}^{\Psi}) + \alpha_{p,a}[X](\tau_{0+}^{\Psi})(\tau - \tau_{0+}^{\Psi}) + \int_{\tau_{0+}^{\Psi}}^{\tau} d\tau'' \int_{\tau_{0+}^{\Psi}}^{\tau''} d\tau' \mathbf{I}_{(3,p)}[\Psi, \mathbf{h}, \pi](\tau') \\
\rho[X](\tau) &= \rho[X](\tau_{0+}^{\Psi})
\end{aligned}$$

Recall (61) and $\sup_{\tau \in \mathcal{I}} |\tau - \tau_{0+}^{\Psi}| \leq 4$. Then we have $d_{\mathcal{E}}(\Psi|_{\mathcal{I}}, \Xi|_{\mathcal{I}}) \leq 2^3 d_{\mathcal{D}}(\Psi(\tau_{0+}^{\Psi}), \Xi(\tau_{0+}^{\Psi})) + 2^{-1} d_{\mathcal{E}}(\Psi|_{\mathcal{I}}, \Xi|_{\mathcal{I}})$. Therefore,

$$(76) \quad d_{\mathcal{E}}(\Psi|_{\mathcal{I}}, \Xi|_{\mathcal{I}}) \leq 2^4 d_{\mathcal{D}}(\Psi(\tau_{0+}^{\Psi}), \Xi(\tau_{0+}^{\Psi})) = 2^4 d_{\mathcal{D}}(\Phi_*(\pi, \mathbf{f}^{\Psi}, \sigma_*), \Phi_*(\pi, \mathbf{f}^{\Upsilon}, \sigma_*)) \xrightarrow{\mathbf{f}^{\Upsilon} \rightarrow \mathbf{f}^{\Psi}} 0$$

Further, we have

$$\begin{aligned}
d_{\mathcal{D}}(\lambda^{\Psi} \Phi_*(\pi', \mathbf{f}^{\Psi}, \sigma_*), \lambda^{\Upsilon} \Phi_*(\pi', \mathbf{f}^{\Upsilon}, \sigma_*)) &= d_{\mathcal{D}}(\Psi(\tau_{2-}^{\Psi}), \Xi(\chi^{-1}(\tau_{2-}^{\Upsilon}))) \\
&\leq d_{\mathcal{D}}(\Psi(\tau_{2-}^{\Psi}), \Psi(\chi^{-1}(\tau_{2-}^{\Upsilon}))) + d_{\mathcal{D}}(\Psi(\chi^{-1}(\tau_{2-}^{\Upsilon})), \Xi(\chi^{-1}(\tau_{2-}^{\Upsilon}))) \\
&\stackrel{(76)}{\leq} d_{\mathcal{D}}(\Psi(\tau_{2-}^{\Psi}), \Psi(\chi^{-1}(\tau_{2-}^{\Upsilon}))) + 2^4 d_{\mathcal{D}}(\Phi_*(\pi, \mathbf{f}^{\Psi}, \sigma_*), \Phi_*(\pi, \mathbf{f}^{\Upsilon}, \sigma_*))
\end{aligned}$$

Now, to prove that Π, Λ and τ_{2-} are continuous, it suffices to show that $\chi^{-1}(\tau_{2-}^{\Upsilon}) \rightarrow \tau_{2-}^{\Psi}$ as $\mathbf{f}^{\Upsilon} \rightarrow \mathbf{f}^{\Psi}$.

Recall that $\alpha_a[\Psi](\tau) \geq \frac{1}{2}$ and $\alpha_{a,a'}[\Psi](\tau) \leq 0$ for all $\tau \in \mathcal{J} \subset \mathcal{J}^{\Psi}$. Therefore, $\forall \tau \in \mathcal{J}$ we have

$$\frac{d}{d\tau}(\xi_a[\Psi, \mathbf{h}^{\Psi}] - \xi_{a'}[\Psi, \mathbf{h}^{\Psi}]) \stackrel{\text{Definition 3.3}}{=} \alpha_a[\Psi] - \alpha_{a'}[\Psi] = 2\alpha_a[\Psi] - \alpha_{a,a'}[\Psi] \geq 1$$

and it follows that $|\xi_a[\Psi, \mathbf{h}^{\Psi}](\tau) - \xi_{a'}[\Psi, \mathbf{h}^{\Psi}](\tau)| \geq |\tau - \tau_{2-}^{\Psi}|$, $\tau \in \mathcal{J}$. Set $\tau = \chi^{-1}(\tau_{2-}^{\Upsilon}) \in \mathcal{J}$. Recall Definition 3.6. Then

$$\begin{aligned} |\chi^{-1}(\tau_{2-}^{\Upsilon}) - \tau_{2-}^{\Psi}| &\leq |\xi_a[\Psi, \mathbf{h}^{\Psi}](\chi^{-1}(\tau_{2-}^{\Upsilon})) - \xi_{a'}[\Psi, \mathbf{h}^{\Psi}](\chi^{-1}(\tau_{2-}^{\Upsilon}))| \\ &\leq 2 \, d_{\mathcal{D}, \mathbf{h}^{\Psi}}(\Psi(\chi^{-1}(\tau_{2-}^{\Upsilon})), \Xi(\chi^{-1}(\tau_{2-}^{\Upsilon}))) \end{aligned}$$

Since $\mathbf{f}^{\Upsilon} \rightarrow \mathbf{f}^{\Psi}$ implies $d_{\mathcal{E}}(\Psi|_{\mathcal{I}}, \Xi|_{\mathcal{I}}) \rightarrow 0$, we have $|\chi^{-1}(\tau_{2-}^{\Upsilon}) - \tau_{2-}^{\Psi}| \rightarrow 0$ as required.

Uniqueness of Π, Λ and τ_{2-} . The argument stays exactly the same as for the vacuum case in [2].

4. ERA-TO-ERA AND EPOCH-TO-EPOCH MAPS

Definition 4.1. (Epoch – to – epochmap) Set $\mathcal{Q}_R : (0, \infty) \setminus \mathbb{Q} \rightarrow (0, \infty) \setminus \mathbb{Q}$

$$w \mapsto \mathcal{Q}_R(w) = \begin{cases} \frac{1}{w} - 1 & \text{if } w < 1 \\ w - 1 & \text{if } w > 1 \end{cases}$$

For every $w \in (0, \infty) \setminus \mathbb{Q}$, set

$$(77) \quad \mathbf{Q}_R\{w\}(q, \mathbf{h}, z) = \left(\frac{\text{num1}}{\text{den}}, \frac{\text{num2}}{\text{den}}, \text{num3} \right)$$

where, if $w < 1$

$$\begin{aligned} \text{num1} &= 1 + w + \mathbf{h} \log 2 - \mathbf{h}(1 + 2w) \log \left(1 + \frac{1}{w} \right) \\ \text{num2} &= \mathbf{h} \\ \text{den} &= (1 + w)(1 + q + \mathbf{h} \log 2) - \mathbf{h}(2 + w) \log \left(1 + \frac{1}{w} \right) \\ \text{num3} &= \left(1 + \frac{1}{w} \right) z \end{aligned}$$

and, if $w > 1$,

$$\begin{aligned} \text{num1} &= (1 + w)(1 + q + \mathbf{h} \log 2) - \mathbf{h}(2 + w) \log \left(1 + \frac{1}{w} \right) \\ \text{num2} &= \mathbf{h}w \\ \text{den} &= 1 + w + \mathbf{h} \log 2 - \mathbf{h}(1 + 2w) \log \left(1 + \frac{1}{w} \right) \\ \text{num3} &= \left(1 + \frac{1}{w} \right) z \end{aligned}$$

Further, $\forall w \in (0, \infty) \setminus \mathbb{Q}$ set

$$\begin{aligned} \mathcal{Q}_R^n &= (\mathcal{Q}_R \circ \dots \circ \mathcal{Q}_R)(w) \\ \mathbf{Q}_R^n\{w\} &= \mathbf{Q}_R\{\mathcal{Q}_R^{n-1}(w)\} \circ \dots \circ \mathbf{Q}_R\{\mathcal{Q}_R^2(w)\} \circ \mathbf{Q}_R\{\mathcal{Q}_R(w)\} \circ \mathbf{Q}_R\{w\} \end{aligned}$$

Definition 4.2. The floor function is $\mathbb{R} \ni x \mapsto \lfloor x \rfloor = \max\{n \in \mathbb{Z} | n \leq x\}$.

Definition 4.3. (Era-to-era map). Define $\mathcal{E}_R : (0, 1) \setminus \mathbb{Q} \rightarrow (0, 1) \setminus \mathbb{Q}$ by $\mathcal{E}_R(w) = \mathcal{Q}_R^{\lfloor 1/w \rfloor}(w)$. For every $w \in (0, 1) \setminus \mathbb{Q}$, denote by $\mathbf{E}_R\{w\}$ the pair of rational functions over \mathbb{R} given by $\mathbf{E}_R\{w\} = \mathbf{Q}_R^{\lfloor 1/w \rfloor}\{w\}$. Finally, for all $w \in (0, 1) \setminus \mathbb{Q}$ and all integers $n \geq 0$, set

$$\begin{aligned}\mathcal{E}^n(w) &= (\mathcal{E}_R \circ \dots \circ \mathcal{E}_R)(w) \\ \mathbf{E}_R^n\{w\} &= \mathbf{E}_R\{\mathcal{E}_R^{n-1}(w)\} \circ \dots \circ \mathbf{E}_R\{\mathcal{E}_R^2(w)\} \circ \mathbf{E}_R\{\mathcal{E}_R(w)\} \circ \mathbf{E}_R\{w\}\end{aligned}$$

Lemma 4.1. *For all integers $m, n \geq 0$,*

- $\mathbf{Q}_R^{m+n}\{w\} = \mathbf{Q}_R^m\{\mathcal{Q}_R^n(w)\} \circ \mathbf{Q}_R^n\{w\}$ for $w \in (0, \infty) \setminus \mathbb{Q}$
- $\mathbf{E}_R^{m+n}\{w\} = \mathbf{E}_R^m\{\mathcal{E}_R^n(w)\} \circ \mathbf{E}_R^n\{w\}$ for $w \in (0, 1) \setminus \mathbb{Q}$

Proof: Consider the equation for $\mathbf{Q}_R^{m+n}\{w\}$. We have

$$\begin{aligned}\text{LHS} &= \mathbf{Q}_R^{m+n}\{w\} \\ &= \mathbf{Q}_R\{\mathcal{Q}_R^{m+n-1}(w)\} \circ \dots \circ \mathbf{Q}_R\{\mathcal{Q}_R(w)\} \circ \mathbf{Q}_R\{w\}\end{aligned}$$

and

$$\begin{aligned}\text{RHS} &= \mathbf{Q}_R^m\{\mathcal{Q}_R^n(w)\} \circ \mathbf{Q}_R^n\{w\} \\ &= \mathbf{Q}_R^m\{\mathcal{Q}_R^n(w)\} \circ (\mathbf{Q}_R\{\mathcal{Q}_R^{n-1}(w)\} \circ \dots \circ \mathbf{Q}_R\{\mathcal{Q}_R^2(w)\} \circ \mathbf{Q}_R\{\mathcal{Q}_R(w)\} \circ \mathbf{Q}_R\{w\}) \\ &= \mathbf{Q}_R\{\mathcal{Q}_R^{m-1}(\mathcal{Q}_R^n(w))\} \circ \dots \circ \mathbf{Q}_R\{\mathcal{Q}_R(\mathcal{Q}_R^n(w))\} \circ \mathbf{Q}_R\{\mathcal{Q}_R^n(w)\} \circ \mathbf{Q}_R\{\mathcal{Q}_R^{n-1}(w)\} \circ \dots \circ \mathbf{Q}_R\{w\} \\ &= \mathbf{Q}_R\{\mathcal{Q}_R^{m+n-1}(w)\} \circ \dots \circ \mathbf{Q}_R\{\mathcal{Q}_R^{1+n}(w)\} \circ \mathbf{Q}_R\{\mathcal{Q}_R^n(w)\} \circ \mathbf{Q}_R\{\mathcal{Q}_R^{n-1}(w)\} \circ \dots \circ \mathbf{Q}_R\{w\}\end{aligned}$$

Analogous statement for $\mathbf{E}_R^{m+n}\{w\}$. \square

Proposition 4.1. *Let $w \in (0, 1) \setminus \mathbb{Q}$. Then, for every integer $1 \leq r \leq \lfloor \frac{1}{w} \rfloor$,*

$$\mathbf{Q}_R^r\{w\}(q, \mathbf{h}, z) = \left(\frac{\text{num}1_r}{\text{den}_r}, \frac{\text{num}2_r}{\text{den}_r}, \text{num}3_r \right)$$

where

$$\begin{aligned}\text{num}1_r &= (1+w)(r+rq-q) + \mathbf{h}A_1(w, r) \\ \text{num}2_r &= \mathbf{h}(w+1-wr) \\ \text{den}_r &= (1+w)(1+q) + \mathbf{h}A_2(w, r) \\ \text{num}3_r &= \frac{1}{1-(r-1)w} \left(1 + \frac{1}{w} \right) z\end{aligned}$$

and where

$$\begin{aligned}
A_1(w, r) &= (2r - 1 + wr - w)\log 2 - (2r - 1 + wr + w)\log \left(1 + \frac{1}{w}\right) \\
&\quad + \sum_{k=1}^{r-1} (1 + 2k - 2k^2w - w)\log \left(1 + \frac{w}{1 - kw}\right) + r \sum_{k=1}^{r-1} ((2k - 1)w - 2)\log \left(1 + \frac{w}{1 - kw}\right) \\
A_2(w, r) &= (1 + wr)\log 2 - (2 + w)\log \left(1 + \frac{1}{w}\right) + r \sum_{k=1}^{r-1} ((2k - 1)w - 2)\log \left(1 + \frac{w}{1 - kw}\right)
\end{aligned}$$

Furthermore, $\mathcal{E}_R(w) = \frac{1}{w} - \lfloor \frac{1}{w} \rfloor$, that is, \mathcal{E}_R is a Gauss map, and

$$(78) \quad \mathbf{E}_R\{w\}(q, \mathbf{h}, z) = \left(\frac{\text{num}1_{\lfloor 1/w \rfloor}}{\text{den}_{\lfloor 1/w \rfloor}}, \frac{\text{num}2_{\lfloor 1/w \rfloor}}{\text{den}_{\lfloor 1/w \rfloor}}, \text{num}3_{\lfloor 1/w \rfloor} \right)$$

Proof (sketch): Let $w \in (0, 1) \setminus \mathbb{Q}$. We use the proof by induction over $1 \leq r \leq \lfloor \frac{1}{w} \rfloor$. The base case $r = 1$ gives by direct substitution exactly $\mathbf{Q}_R\{w\}(q, \mathbf{h})$ from Def.4.1. The induction step is the identity

$$(79) \quad \mathbf{Q}_R\{\mathcal{Q}_R^{r-1}(w)\} \left(\frac{\text{num}1_{r-1}}{\text{den}_{r-1}}, \frac{\text{num}2_{r-1}}{\text{den}_{r-1}}, \text{num}3_{r-1} \right) = \left(\frac{\text{num}1_r}{\text{den}_r}, \frac{\text{num}2_r}{\text{den}_r}, \text{num}3_r \right)$$

for all $2 \leq r \leq \lfloor \frac{1}{w} \rfloor$. We have

$$\begin{aligned}
\lambda \text{num}1_r &= \left(1 + \left(\frac{1}{w} - r + 1\right)\right) (\text{den}_{r-1} + \text{num}1_{r-1} + \text{num}2_{r-1}\log 2) \\
&\quad - \text{num}2_{r-1} \left(2 + \left(\frac{1}{w} - r + 1\right)\right) \log \left(1 + \frac{w}{1 - (r-1)w}\right) \\
\lambda \text{num}2_r &= \text{num}2_{r-1} \left(\frac{1}{w} - r + 1\right) \\
\lambda \text{den}_r &= \text{den}_{r-1} \left(1 + \left(\frac{1}{w} - r + 1\right)\right) + \text{num}2_{r-1}\log 2 \\
&\quad - \text{num}2_{r-1} \left(1 + 2 \left(\frac{1}{w} - r + 1\right)\right) \log \left(1 + \frac{w}{1 - (r-1)w}\right)
\end{aligned}$$

where $\lambda = 2 + \frac{1}{w} - r > 2$.

Further, by equation (77) we have $\left(1 + \frac{1}{\frac{1}{w} - r + 1}\right) \text{num}3_{r-1} = \text{num}3_r$. Therefore, $\lambda \text{num}3_{r-1} = \left(\frac{1}{w} - r + 1\right) \text{num}3_r$.

Equation (78) follows from the definition of $\mathbf{E}_R\{w\}$. \square

Lemma 4.2. *For every $w \in (0, 1) \setminus \mathbb{Q}$ and every integer r with $1 \leq r \leq \lfloor \frac{1}{w} \rfloor$,*

$$\begin{aligned} 0 &\leq A_1(w, r) - rA_2(w, r) + \log 2 \leq 6\frac{1}{w} \\ -8\log\left(1 + \frac{1}{w}\right) &\leq A_2(w, r) \leq 0 \end{aligned}$$

with A_1, A_2 as defined in Proposition 4.1.

Proof. Straightforward. \square

Proposition 4.2. *For every $w \in (0, 1) \setminus \mathbb{Q}$, every $p > 0$ and every integer $1 \leq r \leq \lfloor \frac{1}{w} \rfloor$, let (μ', ν', ζ') be the triple of rational functions over \mathbb{R} in the triple of abstract variables (μ, ν, ζ) given implicitly by*

$$\left(p' + \frac{\mu'}{\nu'}, \frac{1+w'}{\nu'}, \zeta'\right) = \mathbf{Q}_R^r\{w\} \left(p + \frac{\mu}{\nu}, \frac{1+w}{\nu}, \zeta\right)$$

where $w' = \mathbf{Q}_R^r(w) = \frac{1}{w} - r$ and $p' = r - \frac{p}{1+p}$, that is $(p', 0, 0) = \mathbf{Q}_R^r\{w\}(p, 0, 0)$. Then, μ' is actually a linear polynomial over \mathbb{R} in μ , and ν' is a linear polynomial over \mathbb{R} in (μ, ν) , and ζ' is a linear polynomial over \mathbb{R} in ζ . Explicitly

$$(80) \quad \begin{pmatrix} \mu' \\ \nu' \\ \zeta' \end{pmatrix} = \frac{1}{w} \begin{pmatrix} -\frac{1}{1+p} & 0 & 0 \\ 1 & 1+p & 0 \\ 0 & 0 & \frac{1+w}{1+w(1-r)} \end{pmatrix} \begin{pmatrix} \mu \\ \nu \\ \zeta \end{pmatrix} + \frac{1}{w} \begin{pmatrix} A_1(w, r) - p'A_2(w, r) \\ A_2(w, r) \\ 0 \end{pmatrix}$$

The first and the second entries of the vector

$$\frac{1}{w} \begin{pmatrix} A_1(w, r) - p'A_2(w, r) \\ A_2(w, r) \\ 0 \end{pmatrix}$$

are bounded in absolute value by $\leq 2^4 \left(\frac{1}{w}\right)^2$ and $2^3 \frac{1}{w} \log\left(1 + \frac{1}{w}\right)$, respectively.

Remark 4.1. Observe that ζ' is well-defined, that is, for all $1 \leq r \leq \lfloor \frac{1}{w} \rfloor$ we have $\zeta' > 0$ (physical meaning \rightarrow energy density) and $1 + w(1 - r) \neq 0$.

Proof(of Proposition 4.2). Equation (80) follows directly from Proposition 4.1. The bounds follow from Lemma 4.2. \square

Definition 4.4. For every sequence of strictly positive integers $(k_n)_{n \geq 0}$, we denote the associated infinite continued fraction by

$$\langle k_0, k_1, \dots \rangle = \frac{1}{k_0 + \frac{1}{k_1 + \dots}} \in \left(\frac{1}{k_0 + 1}, \frac{1}{k_0} \right) \setminus \mathbb{Q}$$

Every element of $(0, 1) \setminus \mathbb{Q}$ has a unique continued fraction expansion of this form.

Goal: to show that for $\mathbf{h} = 0$, the era-to-era maps can be realized as a left-shift operator on two-sided sequences of positive integers.

Proposition 4.3. Fix $(k_n)_{n \in \mathbb{Z}}$ and define $(p_n)_{n \in \mathbb{Z}}$ and $(w_n)_{n \in \mathbb{Z}}$ by

$$\begin{aligned} \frac{1}{1 + p_n} &= \langle k_n, k_{n-1}, k_{n-2}, \dots \rangle \\ w_n &= \langle k_{n+1}, k_{n+2}, k_{n+3}, \dots \rangle \end{aligned}$$

Then $w_{n+1} = \mathcal{E}_R(w_n)$ and $(p_{n+1}, 0, 0) = \mathbf{E}_R\{w_n\}(p_n, 0, 0)$ for all $n \in \mathbb{Z}$, and $\mathcal{E}_R^n(w_0) = w_n$, and $\mathbf{E}_R^n\{w_0\}(p_0, 0, 0) = (p_n, 0, 0)$ for all $n \geq 0$.

Proof. We have

$$\begin{aligned} \mathcal{E}_R(w_n) &= \frac{1}{w_n} - \left\lfloor \frac{1}{w_n} \right\rfloor \\ &= k_{n+1} + \frac{1}{k_{n+2} + \frac{1}{k_{n+3} + \dots}} - k_{n+1} \\ &= \frac{1}{k_{n+2} + \frac{1}{k_{n+3} + \dots}} \\ &= w_{n+1} \end{aligned}$$

and

$$\begin{aligned} \mathbf{E}_R\{w_n\}(p_n, 0, 0) &= \left(\left\lfloor \frac{1}{w_n} \right\rfloor - 1 + \frac{1}{1 + p_n}, 0, 0 \right) \\ &= (k_{n+1} - 1 + \langle k_n, k_{n-1}, k_{n-2}, \dots \rangle, 0, 0) \\ &= \left(k_{n+1} - 1 + \frac{1}{k_n + \frac{1}{k_{n-1} + \dots}}, 0, 0 \right) \\ &= (p_{n+1}, 0, 0). \end{aligned}$$

□

Definition 4.5. Fix any two-sided sequence $(k_n)_{n \in \mathbb{Z}}$ of strictly positive integers and define $(p_n)_{n \in \mathbb{Z}}$ and $(w_n)_{n \in \mathbb{Z}}$ as in Proposition 4.3. For every integer $n \geq 0$, let (μ_n, ν_n, ζ_n) be the triple of linear polynomials over \mathbb{R} in the abstract variables (μ_0, ν_0, ζ_0) with coefficients depending only on the fixed sequence $(k_n)_{n \in \mathbb{Z}}$, given implicitly by

$$\left(p_n + \frac{\mu_n}{\nu_n}, \frac{1 + w_n}{\nu_n}, \zeta_n\right) = \mathbf{E}_R^n\{w_0\} \left(p_0 + \frac{\mu_0}{\nu_0}, \frac{1 + w_0}{\nu_0}, \zeta_0\right)$$

or by the equivalent recursive prescription

$$(81) \quad \left(p_{n+1} + \frac{\mu_{n+1}}{\nu_{n+1}}, \frac{1 + w_{n+1}}{\nu_{n+1}}, \zeta_{n+1}\right) = \mathbf{E}_R\{w_n\} \left(p_n + \frac{\mu_n}{\nu_n}, \frac{1 + w_n}{\nu_n}, \zeta_n\right)$$

By Proposition 4.2, equation (81) is $V_{n+1} = X_n V_n + Y_n$, where $V_n = (\mu_n, \nu_n, \zeta_n)^T$ and

$$\begin{aligned} X_n &= \frac{1}{w_n} \begin{pmatrix} -\frac{1}{1+p_n} & 0 & 0 \\ 1 & 1+p_n & 0 \\ 0 & 0 & \frac{1+w_n}{1+w_n(1-\lfloor \frac{1}{w_n} \rfloor)} \end{pmatrix} \\ Y_n &= \frac{1}{w_n} \begin{pmatrix} A_1(w_n) - p_{n+1}A_2(w_n) \\ A_2(w_n) \\ 0 \end{pmatrix} \end{aligned}$$

Here, $A_1(w) = A_1(w, \lfloor \frac{1}{w} \rfloor)$ and $A_2(w) = A_2(w, \lfloor \frac{1}{w} \rfloor)$.

Example 4.1 Consider Definition 4.5 when $k_n = 1$ for all $n \in \mathbb{Z}$. Then $w_n = p_n = w = \frac{1}{2}(\sqrt{5} - 1) \in (0, 1) \setminus \mathbb{Q}$, for all $n \in \mathbb{Z}$. We have $\lfloor \frac{1}{w} \rfloor = 1$ and, therefore,

$$\begin{aligned} X_n &= \begin{pmatrix} -1 & 0 & 0 \\ 1+w & 2+w & 0 \\ 0 & 0 & 1+w \end{pmatrix} \\ Y_n &= \begin{pmatrix} -2\log(1+w) \\ (2+w)\log 2 - (6+4w)\log(1+w) \\ 0 \end{pmatrix} \end{aligned}$$

for all $n \geq 0$. It follows that

$$\begin{aligned}
\mu_{n+1} &= -\mu_n - 2\log(1+w) \\
\nu_{n+1} &= (1+w)\mu_n + (2+w)\nu_n + (2+w)\log 2 - (6+4w)\log(1+w) \\
\zeta_{n+1} &= (1+w)\zeta_n
\end{aligned}$$

and we have $\mu_{n+2} = \mu_n$ for all $n \geq 0$, which implies $\mu_{2n} = \mu_0$ and $\mu_{2n+1} = -\mu_0 - 2\log(1+w)$. We can, therefore, identify unique $\lambda_1 = \lambda_1(\mu_0)$ and $\lambda_2 = \lambda_2(\mu_0)$, such that

$$\begin{aligned}
\nu_{2n} &= (2+w)^{2n}(\nu_0 - \lambda_1) + \lambda_1 \\
\nu_{2n+1} &= (2+w)^{2n}(\nu_1 - \lambda_2) + \lambda_2.
\end{aligned}$$

Definition 4.6. (Propagator). Let $(p_n)_{n \in \mathbb{Z}}$, $(w_n)_{n \in \mathbb{Z}}$, $(X_n)_{n \geq 0}$ be as in Definition 4.5. Then for all integers $n \geq m \geq 0$, let $P_{n,m} = X_{n-1} \cdots X_m$. Explicitly,

$$P_{n,m} = \begin{pmatrix} a_{n-1} \cdots a_m & 0 & 0 \\ \sum_{l=m}^{n-1} x_l & c_{n-1} \cdots c_m & 0 \\ 0 & 0 & d_{n-1} \cdots d_m \end{pmatrix}$$

where $x_l = c_{n-1} \cdots c_{l+1} b_l a_{l-1} \cdots a_m$ whenever $n-1 \geq l \geq m$, and for all $l \geq 0$,

$$X_l = \begin{pmatrix} a_l & 0 & 0 \\ b_l & c_l & 0 \\ 0 & 0 & d_l \end{pmatrix} \quad a_l = \frac{-1}{w_l(1+p_l)} \quad b_l = \frac{1}{w_l} \quad c_l = \frac{1+p_l}{w_l} \quad d_l = \frac{1}{w_l} \frac{1+w_l}{1+w_l \left(1 - \lfloor \frac{1}{w_l} \rfloor\right)}$$

In this definition, a sequence of dots \cdots indicates that indices increase towards the left, one by one. A product of the form $F_k \cdots F_j$ is equal to one if $k = j - 1$. In particular, $P_{n,n} = \mathbb{I}_3$.

Lemma 4.3. Recall Definition 4.5. We have

$$V_n = P_{n,0}V_0 + \sum_{l=0}^{n-1} P_{n,l+1}Y_l$$

Proof: By direct inspection using Definition 4.5. Namely,

$$\begin{aligned}
V_n &= X_{n-1}V_{n-1} + Y_{n-1} \\
&= X_{n-1}(X_{n-2}V_{n-2} + Y_{n-2}) + Y_{n-1} \\
&= X_{n-1}X_{n-2}V_{n-2} + X_{n-1}Y_{n-2} + Y_{n-1} \\
&= X_{n-1}X_{n-2}(X_{n-3}V_{n-3} + Y_{n-3}) + X_{n-1}Y_{n-2} + Y_{n-1} \\
&= X_{n-1}X_{n-2}X_{n-3}V_{n-3} + X_{n-1}X_{n-2}Y_{n-3} + X_{n-1}Y_{n-2} + Y_{n-1} \\
&= \dots \\
&= P_{n,0}V_0 + \sum_{l=0}^{\infty} P_{n,l+1}Y_l
\end{aligned}$$

□

Lemma 4.4. *Recall Definition 4.6. For all integers $n \geq m \geq 0$, we have*

$$\begin{aligned}
\frac{1}{2} &\leq \frac{w_{n-1}}{w_{m-1}}(-1)^{m+n}a_{n-1} \cdots a_m && \leq 2 \\
(1 - \delta_{mn})\frac{1}{4} &\leq \frac{w_{n-1}}{w_{m-1}^2}(w_{n-2} \cdots w_{m-1})^2 \sum_{l=m}^{n-1} x_l && \leq 2 \\
\frac{1}{2} &\leq \frac{w_{n-1}}{w_{m-1}}(w_{n-2} \cdots w_{m-1})^2 c_{n-1} \cdots c_m && \leq 2 \\
(82) \quad \frac{1}{2} &< \frac{w_{n-1}^2}{w_{m-1}^2}(w_{n-2} \cdots w_{m-1})^2 d_{n-1} \cdots d_m && \leq 2
\end{aligned}$$

Proof: Only for the new product $d_{n-1} \cdots d_m$. Recall $n \geq m$ so in the sequence $d_{n-1} \cdots d_m$ the indices are increasing towards the left. For the other three stays exactly the same as for the vacuum. Recall Definition 4.6. Recall Proposition 4.3. We have $w_{n+1} = \mathcal{E}_R(w_n)$. Therefore,

$$\begin{aligned}
1 + w_l \left(1 - \lfloor \frac{1}{w_l} \rfloor \right) &= 1 + w_l \left(1 - \frac{1}{w_l} + w_{l+1} \right) \\
&= w_l(1 + w_{l+1})
\end{aligned}$$

and we get

$$\begin{aligned}
d_{n-1} \cdots d_m &= \frac{1}{w_{n-2} \cdots w_{m-1}} \frac{w_{m-1}}{w_{n-1}} \frac{(1+w_{n-1}) \cdots (1+w_m)}{w_{n-1}(1+w_n)w_{n-2}(1+w_{n-1}) \cdots w_m(1+w_{m+1})} \\
&= \frac{1}{w_{n-2} \cdots w_{m-1}} \frac{w_{m-1}}{w_{n-1}} \frac{1+w_m}{1+w_n} \frac{1}{w_{n-1}w_{n-2} \cdots w_m} \\
&= \frac{1}{w_{n-2} \cdots w_{m-1}} \frac{w_{m-1}}{w_{n-1}} \frac{1+w_m}{1+w_n} \frac{w_{m-1}}{w_{n-1}} \frac{1}{w_{n-2} \cdots w_{m-1}} \\
&= \left(\frac{1}{w_{n-2} \cdots w_{m-1}} \right)^2 \frac{w_{m-1}^2}{w_{n-1}^2} \frac{1+w_m}{1+w_n}
\end{aligned}$$

Therefore,

$$\frac{1}{2} \leq \frac{w_{n-1}^2}{w_{m-1}^2} (w_{n-2} \cdots w_{m-1})^2 d_{n-1} \cdots d_m \leq 2$$

□

Proposition 4.4. *For all $w_0 \in (0, 1) \setminus \mathbb{Q}$ and $q_0 \in (0, \infty) \setminus \mathbb{Q}$, introduce*

- *a two sided sequence of strictly positive integers $(k_n)_{n \in \mathbb{Z}}$ by*

$$\begin{aligned}
(1+q_0)^{-1} &= \langle k_0, k_{-1}, k_{-2}, \dots \rangle \\
w_0 &= \langle k_1, k_2, k_3, \dots \rangle
\end{aligned}$$

- *(Era Pointer) $J : \mathbb{Z} \rightarrow \mathbb{Z}$ by $J(0) = 0$ and $J(n+1) = J(n) + k_{n+1}$*
- *(Era Counter) $N : \mathbb{Z} \rightarrow \mathbb{Z}$ by $N(0) = 0$ and $N(j+1) = N(j) + \chi_{J(\mathbb{Z})}(j)$, with χ -characteristic function*
- *sequences $(w_j)_{j \in \mathbb{Z}}$ and $(p_j)_{j \in \mathbb{Z}}$ by*

$$\begin{aligned}
w_j &= \langle k_{N(j)+1}, k_{N(j)+2}, \dots \rangle + J(N(j)) - j \\
p_j &= \langle k_{N(j)-1}, k_{N(j)-2}, \dots \rangle + k_{N(j)} + j - J(N(j)) - 1
\end{aligned}$$

Part 1. *Then $p_0 = q_0$ and $w_j, p_j > 0$ and $\mathcal{Q}_R(w_j) = w_{j+1}$ and $\mathbf{Q}_R\{w_j\}(p_j, 0, 0) = (p_{j+1}, 0, 0)$ for all $j \in \mathbb{Z}$, and $\mathcal{Q}_R^j(w_0) = w_j$ and $\mathbf{Q}_R^j\{w_0\}(q_0, 0, 0) = (p_j, 0, 0)$ for all $j \geq 0$.*

Part 2. *Introduce ρ_+ and $\mathbf{C}_0 = \mathbf{C}_0(w_0, q_0)$. Suppose $\mathbf{C}_0(w_0, q_0) < \infty$. Fix \mathbf{h}_0 in the*

interval $0 < \mathbf{h}_0 < \mathbf{C}_0(w_0, q_0)$.

Then there are sequences $(q_j)_{j \geq 0}$, $(\mathbf{h}_j)_{j \geq 0}$, $(z_j)_{j \geq 0}$ of real numbers such that for every $j \geq 0$

$$(q_{j+1}, \mathbf{h}_{j+1}, z_{j+1}) = \mathbf{Q}_R\{w_j\}(q_j, \mathbf{h}_j, z_j)$$

or $(q_j, \mathbf{h}_j, z_j) = \mathbf{Q}_R^j\{w_0\}(q_0, \mathbf{h}_0, z_0)$. For all $j \geq 0$,

- $0 < \mathbf{h}_j \leq 2^6 \mathbf{h}_0 \rho_+^{-2N(j)}$ and

$$\frac{1}{4} \leq \frac{\mathbf{h}_j}{\mathbf{h}_0} \frac{1+w_0}{1+w_j} \prod_{l=0}^{N(j)-1} \frac{1}{w_{J(l)} w_{J(l-1)}} \leq 4$$

- $0 < z_j \leq 2^3 z_0 \rho_+^{-2N(j)}$ and

$$\frac{1}{2} \leq \frac{z_j}{z_0} \prod_{l=1}^{N(j)-1} w_{J(l)} w_{J(l-1)} \leq 2$$

- $q_j \in (0, \infty) \setminus \mathbb{Z}$ and $|q_j - p_j| \leq 2^{12} \mathbf{h}_0 N(j) \rho_+^{-2N(j)} k_{N(j)}$
- $q_j \in (0, 1)$ if and only if $p_j \in (0, 1)$ if and only if $j - 1 \in J(\mathbb{Z})$
- $\max\{\frac{1}{w_j}, \frac{1}{q_j}, \frac{1}{|q_j-1|}, q_j, z_j\} \leq 2^4 \max\{k_{N(j)-2}, k_{N(j)-1}, k_{N(j)} k_{N(j)+1}\}$

Part 3. Let the map $\mathcal{Q}_L : (0, \infty)^4 \rightarrow (0, \infty)^2 \times \mathbb{R} \times (0, \infty)$ be given as in Definition 3.16. Then the sequences $(\mathbf{h}_j)_{j \geq 0}$, $(w_j)_{j \geq 0}$, $(q_j)_{j \geq 0}$, $(z_j)_{j \geq 0}$ in Part 2 satisfy for all $j \geq 0$:

$$(\mathbf{h}_j, w_j, q_j, z_j) = \mathcal{Q}_L(\mathbf{h}_{j+1}, w_{j+1}, q_{j+1}, z_{j+1})$$

Proof of Part 1. The two basic properties of J and N remain the same as for the vacuum case. That is, for all $j \in \mathbb{Z}$:

- $N \circ J$ is the identity and, therefore, $J(N(j)) = j \Leftrightarrow j \in J(\mathbb{Z})$
- $J(N(j)) \geq j$ and $J(N(j) - 1) \leq j - 1$ by definition of *Era Counter*. Therefore,

$$(83) \quad j \leq J(N(j)) \leq k_{N(j)} + j - 1$$

The second bullet implies that $J(N(j)) - j \geq 0$ and, therefore, $w_j > 0 \forall j \in \mathbb{Z}$. Also, it implies that $k_{N(j)} + j - 1 - J(N(j)) \geq 0$ and, therefore, $p_j > 0 \forall j \in \mathbb{Z}$.

The first bullet implies that $w_j \in (0, 1) \Leftrightarrow j \in J(\mathbb{Z})$ (follows directly from definition of w_j).

Then we have

$$\begin{aligned}\mathcal{Q}_R(w_j) &= \begin{cases} \frac{1}{w_j} - 1 & \text{if } j \in J(\mathbb{Z}) \\ w_j - 1 & \text{if } j \notin J(\mathbb{Z}) \end{cases} \\ \mathbf{Q}_R\{w_j\}(p_j, 0, 0) &= \begin{cases} \left(\frac{1}{1+p_j}, 0, 0\right) & \text{if } j \in J(\mathbb{Z}) \\ (1 + p_j, 0, 0) & \text{if } j \notin J(\mathbb{Z}) \end{cases}\end{aligned}$$

Proof of Part 2. We first construct sequences $(q_j)_{j \geq 0}$, $(\mathbf{h}_j)_{j \geq 0}$, and $(z_j)_{j \geq 0}$. Then we verify that they have the desired properties. A sequence of dots \cdots indicates that indices increase towards the left, one by one. A product of the form $F_m \cdots F_n$ is equal to one if $m = n - 1$.

Define sequences $(w_n^*)_{n \in \mathbb{Z}}$ and $(p_n^*)_{n \in \mathbb{Z}}$ by $w_n^* = w_{J(n)} \in (0, 1) \setminus \mathbb{Q}$ and $p_n^* = p_{J(n)} \in (0, \infty) \setminus \mathbb{Q}$. Using the results from Part 1, we can write

$$\begin{aligned}\frac{1}{1 + p_n^*} &= \langle k_n, k_{n-1}, k_{n-2} \cdots \rangle \\ w_n^* &= \langle k_{n+1}, k_{n+2}, k_{n+3} \cdots \rangle\end{aligned}$$

Proposition 4.3 implies that for any such p_n^* and w_n^* we have $w_{n+1}^* = \mathcal{E}_R(w_n^*)$ and $(p_{n+1}^*, 0, 0) = \mathbf{E}_R\{w_n^*\}(p_n^*, 0, 0)$.

Recall Definition 4.5. Let $V_n^* = (\mu_n^*, \nu_n^*, \zeta_n^*)^T$ be the solution to $V_{n+1}^* = X_n^* V_n^* + Y_n^*$ $\forall n \geq 0$ with $\mu_0^* = 0$, $\nu_0^* = \frac{1+w_0^*}{\mathbf{h}_0} > 0$, and $\zeta_0^* = z_0$. Further, we have

$$\begin{aligned}X_n^* &= \frac{1}{w_n^*} \begin{pmatrix} -\frac{1}{1+p_n^*} & 0 & 0 \\ 1 & 1+p_n^* & 0 \\ 0 & 0 & \frac{1+w_n^*}{1+w_n^*(1-\lfloor \frac{1}{w_n^*} \rfloor)} \end{pmatrix} \\ Y_n^* &= \frac{1}{w_n^*} \begin{pmatrix} A_1(w_n^*) - p_{n+1}^* A_2(w_n^*) \\ A_2(w_n^*) \\ 0 \end{pmatrix}\end{aligned}$$

Let $V_j = (\mu_j, \nu_j, \zeta_j)^T$ be given by $V_0 = V_0^*$ and $V_j = X_{N(j)-1}^* V_{N(j)-1}^* + Y_j$, $\forall j \geq 1$, with

$$Y_j = \frac{1}{w_s^*} \begin{pmatrix} A_1(w_s^*, j - J(s)) - p_j A_2(w_s^*, j - J(s)) \\ A_2(w_s^*, j - J(s)) \\ 0 \end{pmatrix}$$

for $s = N(j) - 1$. The functions A_1, A_2 are well-defined because $A_1(w) = A_1(w, \lfloor \frac{1}{w} \rfloor)$ (as

given with only one argument in the expressions for X and Y) and $1 \leq j - J(s) \leq k_{N(j)} = \lfloor \frac{1}{w_s^*} \rfloor$. Both inequalities are the direct consequence of the second bullet in the Proof of Part 1. The last equality is just the definition of w_s^* .

(two observations about Y_j stay the same as for the vacuum case since the third component is zero).

Recall Definition 4.6. Set $P_{n,m}^* = X_{n-1}^* \cdots X_m^*$ for all $n \geq m \geq 0$. By Lemma 4.3 we have

$$V_j = X_s^*(P_{s,0}^* V_0^* + \sum_{l=0}^{s-1} P_{s,l+1}^* Y_l^*) + Y_j = P_{s+1,0}^* V_0^* + \sum_{l=0}^{s-1} P_{s+1,l+1}^* Y_l^* + Y_j$$

which implies

$$\begin{aligned} \mu_j &= a_s^* \cdots a_0^* \mu_0^* + \sum_{l=0}^{s-1} a_s^* \cdots a_{l+1}^* \frac{1}{w_l^*} (A_1(w_l^*) - p_{l+1}^* A_2(w_l^*)) + \frac{1}{w_j} (A_1(w_j) - p_{j+1} A_2(w_j)) \\ &\stackrel{\mu_0^*=0}{=} \sum_{l=0}^{s-1} a_s^* \cdots a_{l+1}^* \frac{1}{w_l^*} (A_1(w_l^*) - p_{l+1}^* A_2(w_l^*)) + \frac{1}{w_j} (A_1(w_j) - p_{j+1} A_2(w_j)) \\ \nu_j &= \mu_0^* \sum_{l=0}^s x_l + c_s^* \cdots c_0^* \nu_0^* + \sum_{l=0}^{s-1} \left(\sum_{k=l+1}^s x_k \frac{1}{w_l^*} (A_1(w_l^*) - p_{l+1}^* A_2(w_l^*)) + c_s^* \cdots c_{l+1}^* \frac{1}{w_l^*} A_2(w_l^*) \right) \\ \zeta_j &= d_s^* \cdots d_0^* \zeta_0^* \end{aligned}$$

Therefore, using Lemma 4.4 (Version 2)

$$|\zeta_j| = |d_s^* \cdots d_0^*| |\zeta_0^*| \leq 2 \left(\frac{1}{w_{s-1}^* \cdots w_{-1}^*} \right)^2 \left(\frac{w_{-1}^*}{w_s^*} \right)^2 |\zeta_0^*|$$

and

$$|\zeta_j| = |d_s^* \cdots d_0^*| |\zeta_0^*| \geq \frac{1}{2} \left(\frac{1}{w_{s-1}^* \cdots w_{-1}^*} \right)^2 \left(\frac{w_{-1}^*}{w_s^*} \right)^2 |\zeta_0^*|$$

and the old estimates for $|\mu_j|$ and ν_j stay the same (checked), that is

$$\begin{aligned}
|\mu_j| &\leq \frac{2^5}{w_s^*} \sum_{l=0}^s \frac{1}{w_l^*} \\
\nu_j &\geq \frac{1}{2w_s^*} \left(\frac{1}{w_{s-1}^* \cdots w_{-1}^*} \right)^2 \left(w_{-1}^* \nu_0 - 2^8 \sum_{l=0}^s (w_{l-1}^* \cdots w_{-1}^*)^2 \log \left(1 + \frac{1}{w_l^*} \right) \right) \\
\nu_j &\leq \frac{2}{w_s^*} \left(\frac{1}{w_{s-1}^* \cdots w_{-1}^*} \right)^2 \left(w_{-1}^* \nu_0 + 2^6 \sum_{l=0}^s (w_{l-1}^* \cdots w_{-1}^*)^2 \log \left(1 + \frac{1}{w_l^*} \right) \right)
\end{aligned}$$

for all $j \geq 1$ and $s = N(j) - 1$. All four estimates are also true when $j = 0, s = -1$. Now, using the (old) estimate $2^8 \sum_{l=0}^s (w_{l-1}^* \cdots w_{-1}^*)^2 \log(1 + 1/w_l^*) \leq 2^{-1} w_{-1}^* \frac{1}{\mathbf{h}_0}$ (checked), we have by direct calculation

$$(84) \quad \frac{1}{4} \leq \frac{w_{N(j)-1}^*}{w_{-1}^*} (w_{N(j)-2}^* \cdots w_{-1}^*)^2 \frac{\mathbf{h}_0}{1 + w_0^*} \nu_j \leq 4$$

Further, we estimate

$$(85) \quad \frac{1}{2} \leq \left(\frac{w_{N(j)-1}^*}{w_{-1}^*} \right)^2 (w_{N(j)-2}^* \cdots w_{-1}^*)^2 \frac{1}{\zeta_0^*} \zeta_j \leq 2$$

Introduce $\mathcal{Z}_j = z_0 \prod_{l=1}^{N(j)-1} \frac{1}{w_{J(l)} w_{J(l-1)}}$. Therefore, we have

$$\frac{1}{2} \leq \frac{z_j}{\mathcal{Z}_j} \leq 2$$

□.

5. ABSTRACT SEMI-GLOBAL EXISTENCE THEOREM

Stays exactly the same as for the vacuum case. See [2].

6. MAIN THEOREMS

Definition 6.1. Let $\|\cdot\|$ be the Euclidean distance in \mathbb{R}^4 . For every $\delta > 0$ and every $\mathbf{f} \in \mathbb{R}^4$, set $B[\delta, \mathbf{f}] = \{\mathbf{g} \in \mathbb{R}^4 \mid \|\mathbf{g} - \mathbf{f}\| \leq \delta\}$

Definition 6.2. Let $\mathcal{F} \subset (0, \infty)^4$ be as in Definition 3.19. For all $\zeta \geq 1$ set

$$B_\zeta \mathcal{F} = \{(\delta, \mathbf{f}) \in [0, \infty) \times \mathcal{F} \mid B[\zeta \delta, \mathbf{f}] \subset \mathcal{F}\} \quad \text{and} \quad B\mathcal{F} = B_1 \mathcal{F}$$

Lemma 6.1. *For all $(\delta, \mathbf{f}) \in B\mathcal{F}$ set*

$$\begin{aligned} W(\delta, \mathbf{f}) &= \max\left\{\frac{1}{w-\delta}, w+\delta, \frac{1}{q-\delta}, \frac{1}{|q-1|-\delta}, q+\delta\right\} && \in [1, \infty) \\ W(\mathbf{f}) = W(0, \mathbf{f}) &= \max\left\{\frac{1}{w}, w, \frac{1}{q}, \frac{1}{|q-1|}, q\right\} && \in [1, \infty) \end{aligned}$$

where $\mathbf{f} = (\mathbf{h}, w, q, z)$. Then

- (1) $W(\mathbf{g}) \leq W(\delta, \mathbf{f})$ for all $\mathbf{g} \in B[\delta, \mathbf{f}]$.
- (2) If $(\delta, \mathbf{f}) \in B_2\mathcal{F} \subset B\mathcal{F}$ then $W(\delta, \mathbf{f}) \leq 2W(\mathbf{f})$.

Proof: straightforward by direct substitution. \square

Lemma 6.2. *Let $\text{Err} : B\mathcal{F} \rightarrow [0, \infty)$ be given by*

$$\text{Err}(\delta, \mathbf{f}) = 2^{42} \left(\frac{1}{\mathbf{h} - \delta} \right)^2 W(\delta, \mathbf{f})^5 \exp \left(-\frac{1}{\mathbf{h}} 2^{-9} W(\delta, \mathbf{f})^{-2} \right) \max\{1, z\}^2$$

where $\mathbf{f} = (\mathbf{h}, w, q, z)$. Then for all $(\delta, \mathbf{f}) \in B\mathcal{F}$, we have $\mathbf{K}(\mathbf{g}) \leq \text{Err}(\delta, \mathbf{f})$ for all $\mathbf{g} \in B[\delta, \mathbf{f}] \subset \mathcal{F}$.

Proof: Let $\mathbf{g} = (\mathbf{h}', w', q', z') \in B[\delta, \mathbf{f}]$. Then, by $\tau_* \geq \mathbf{X}(-1, 0, 0, -1, -1, 0, 0, 0)$ in the notation of Proposition 3.3, we have $\tau_*(\mathbf{g}) \geq \frac{1}{2}W(\mathbf{g})^{-2}$. Further, $0 < x - \delta \leq x' \leq x + \delta \leq 2x$ for $x \in \{\mathbf{h}, z\}$, which obviously implies $\frac{1}{\mathbf{h}'} \geq \frac{1}{2\mathbf{h}}$ and $\max\{1, z'\}^2 \leq 2^2 \max\{1, z\}^2$. Recall Definition 3.18, and it follows

$$\begin{aligned} \mathbf{K}(\mathbf{g}) &\leq 2^{42} \left(\frac{1}{\mathbf{h} - \delta} \right)^2 W(\mathbf{g})^5 \exp \left(-\frac{1}{\mathbf{h}} 2^{-9} W(\mathbf{g})^{-2} \right) \max\{1, z\}^2 \\ &\stackrel{\text{Lemma 6.1, (1)}}{\leq} 2^{42} \left(\frac{1}{\mathbf{h} - \delta} \right)^2 W(\delta, \mathbf{f})^7 \exp \left(-\frac{1}{\mathbf{h}} 2^{-9} W(\delta, \mathbf{f})^{-2} \right) \max\{1, z\}^2 \end{aligned}$$

\square

Lemma 6.3. *Let \mathcal{Q}_L be as in Definition 3.16. Set $\text{Lip} : B\mathcal{F} \rightarrow [0, \infty)$, $\text{Lip}(\delta, \mathbf{f}) = 2^{13}W(\delta, \mathbf{f})^3 \max\{1, z\}$. Then $\|\mathcal{Q}_L(\mathbf{g}) - \mathcal{Q}_L(\mathbf{g}')\| \leq \text{Lip}(\delta, \mathbf{f})\|\mathbf{g} - \mathbf{g}'\|$ for all $\mathbf{g}, \mathbf{g}' \in B[\delta, \mathbf{f}]$.*

Proof. Let $\mathbf{f} = (\mathbf{h}, w, q, z)$. Recall NewLemma B.1 of Appendix B (result summary: the same result as for the old case just multiplied by $\max\{1, z\}$ due to $|dz_L/dw|$ derivative). The case $\mathbf{g} = \mathbf{g}'$ is trivial. Suppose $\mathbf{g} \neq \mathbf{g}'$. Then identify \mathbf{f}_1 and \mathbf{f}_2 in the formulation of NewLemma B.1 with $\mathbf{f}_1 = (\mathbf{h}_1, w_1, q_1, z_1) = \mathbf{g}$ and $\mathbf{f}_2 = (\mathbf{h}_2, w_2, q_2, z_2) = \mathbf{g}'$.

Observe $0 < \mathbf{h}_i \leq 1$ for $i = 1, 2$, by $\mathbf{g}, \mathbf{g}' \in B[\delta, \mathbf{f}] \subset \mathcal{F}$.

$\delta < |q - 1| \Rightarrow$ either $q, q_1, q_2 < 1$ or $q, q_1, q_2 > 1$.

Using $w_{\max} \leq \max\{W(\mathbf{g}), W(\mathbf{g}')\}$, $q_{\max} \leq \max\{W(\mathbf{g}), W(\mathbf{g}')\}$, and $q_{\min}^{-1} = \max\{q_1^{-1}, q_2^{-1}\} \leq$

$\max\{W(\mathbf{g}), W(\mathbf{g}')\}$, we estimate

$$\begin{aligned} 2^{12} q_{\min}^{-2} \log(2 + w_{\max}) \max\{1, z\} &\leq 2^{12} W(\delta, \mathbf{f})^2 (1 + w_{\max}) \max\{1, z\} \\ &\leq 2^{12} W(\delta, \mathbf{f})^2 (1 + W(\delta, \mathbf{f})) \max\{1, z\} \\ &\leq 2^{13} W(\delta, \mathbf{f})^3 \max\{1, z\} \end{aligned}$$

□

Theorem 6.1. *Recall the definitions of $\mathcal{P}_L, \mathcal{Q}_L, \mathcal{F}, \Pi$ from Section 3, Definition 6.2, Lemma 6.1, Lemma 6.2 and Lemma 6.3. Suppose:*

- (1) $(\mathbf{f}_j)_{j \geq 0}$, with $\mathbf{f}_j = (\mathbf{h}_j, w_j, q_j, z_j) \in \mathcal{F}$ satisfies $\mathbf{f}_{j-1} = \mathcal{Q}_L(\mathbf{f}_j)$ for all $j \geq 1$.
- (2) The sequence $(\delta_j)_{j \geq 0}$ given by

$$\delta_j = \sum_{l=j+1}^{\infty} \left\{ \prod_{k=j+1}^{l-1} 2^{16} W(\mathbf{f}_k)^3 \max\{1, z_k\} \right\} 2^{49} \left(\frac{1}{\mathbf{h}_l} \right)^2 W(\mathbf{f}_l)^5 \exp \left(-\frac{1}{\mathbf{h}_l} 2^{-11} W(\mathbf{f}_l)^{-2} \right) \max\{1, z_l\}^2$$

satisfies $\delta_j < \infty$ and $(\delta_j, \mathbf{f}_j) \in B_2 \mathcal{F}$ for all $j \geq 0$.

- (3) $\pi_0 \in S_3$ and $(\pi_j)_{j \geq 0}$ is the unique sequence in S_3 that satisfies $\pi_{j-1} = \mathcal{P}_L(\pi_j, \mathbf{f}_j)$ for all $j \geq 1$.
- (4) $\sigma \in \{-1, +1\}^3$.

Then, there exists a sequence $(\mathbf{g}_j)_{j \geq 0}$ with $\mathbf{g}_j \in B[\delta_j, \mathbf{f}_j] \subset \mathcal{F}$ such that for all $j \geq 1$:

$$\mathbf{g}_{j-1} = \Pi[\pi_j, \sigma_*](\mathbf{g}_j) \quad \text{and} \quad \pi_{j-1} = \mathcal{P}_L(\pi_j, \mathbf{g}_j)$$

Proof: For the proof of this theorem, we use exclusively Proposition 5.1. The abstract object of the latter proposition are in our case identified with

$$\begin{aligned} d &\rightarrow 4 \\ \mathcal{F} &\rightarrow \mathcal{F} \text{ as in Definition 3.19} \\ \Pi_j &\rightarrow \Pi[\pi_j, \sigma_*], \text{ see Proposition 3.3 and hypotheses (3) and (4) of Theorem 6.1} \\ \mathcal{Q}_L &\rightarrow \mathcal{Q}_L|_{\mathcal{F}}, \text{ with } \mathcal{Q}_L \text{ as in Definition 3.16} \\ \text{Err} &\rightarrow \text{Err as in Lemma 6.2} \\ \text{Lip} &\rightarrow \text{Lip as in Lemma 6.3} \\ (\delta_j, \mathbf{f}_j) &\rightarrow (\delta_j, \mathbf{f}_j) \text{ as in hypotheses (1) and (2) of Theorem 6.1} \end{aligned}$$

We now check that the assumptions of the Proposition 5.1 are actually satisfied.

- (a) Consistent with Definition 6.2.
- (b) $\Pi[\pi_j, \sigma_*] : \mathcal{F} \rightarrow (0, \infty)^2 \times \mathbb{R} \times (0, \infty) \subset \mathbb{R}^4$ is continuous by Proposition 3.3.
- (c) Consistent with Err and Lip from Lemma 6.2 and Lemma 6.3.
- (d) Recall Lemma 6.1, (2). By direct calculation

$$\begin{aligned}
\sum_{l=j+1}^{\infty} \left\{ \prod_{k=j+1}^{l-1} \text{Lip}(\delta_k, \mathbf{f}_k) \right\} \text{Err}(\delta_l, \mathbf{f}_l) &= \sum_{l=j+1}^{\infty} \left\{ \prod_{k=j+1}^{l-1} 2^{13} W(\delta_k, \mathbf{f}_k)^3 \max\{1, z_k\} \right\} 2^{42} \left(\frac{1}{\mathbf{h}_l - \delta_l} \right)^2 W(\delta_l, \mathbf{f}_l)^5 \\
&\times \exp \left(-\frac{1}{\mathbf{h}_l} 2^{-9} W(\delta_l, \mathbf{f}_l)^{-2} \right) \max\{1, z_l\}^2 \\
&\leq \sum_{l=j+1}^{\infty} \left\{ \prod_{k=j+1}^{l-1} 2^{16} W(\mathbf{f}_k)^3 \max\{1, z_k\} \right\} 2^{49} \left(\frac{1}{\mathbf{h}_l} \right)^2 W(\mathbf{f}_l)^5 \\
&\times \exp \left(-\frac{1}{\mathbf{h}_l} 2^{-11} W(\mathbf{f}_l)^{-2} \right) \max\{1, z_l\}^2 \\
&= \delta_j.
\end{aligned}$$

□

Theorem 6.2. *Recall Proposition 4.4. Suppose the vector $\mathbf{f}_0 = (\mathbf{h}_0, w_0, q_0, z_0)$ satisfies the following assumptions:*

$$\begin{array}{ll}
w_0 \in (0, 1) \setminus \mathbb{Q} & \mathbf{C}(w_0, q_0) < \infty \\
q_0 \in (0, \infty) \setminus \mathbb{Q} & 0 < \mathbf{h}_0 \leq 2^{-14} (\mathbf{C}(w_0, q_0))^{-1} \\
z_0 > 0 &
\end{array}$$

Recall the definitions of the Era Pointer, Era Counter, and of the sequences $(k_n)_{n \in \mathbb{Z}}$, $(w_j)_{j \in \mathbb{Z}}$, $(q_j)_{j \geq 0}$, $(\mathbf{h}_j)_{j \geq 0}$, $(z_j)_{j \geq 0}$ from the Proposition 4.4. Introduce the sequence $(\mathbf{f}_j)_{j \geq 0}$ by

$$\mathbf{f}_j = (\mathbf{h}_j, w_j, q_j, z_j) \in (0, \infty)^4$$

Introduce sequences $(\mathbf{H}_j)_{j \geq 0}$, $(K_j)_{j \geq 0}$ and $(\mathcal{Z}_j)_{j \geq 0}$ by

$$\begin{aligned}
\mathbf{H}_j &= \mathbf{h}_0 \frac{1+w_j}{1+w_0} \prod_{l=0}^{N(j)-1} w_{J(l)} w_{J(l-1)} &> 0 \\
K_j &= \max\{k_{N(j)-2}, k_{N(j)-1}, k_{N(j)}, k_{N(j)+1}\} &\geq 1 \\
\mathcal{Z}_j &= z_0 \prod_{l=1}^{N(j)-1} \frac{1}{w_{J(l)} w_{J(l-1)}} &> 0
\end{aligned}$$

Suppose:

- (1) $\mathbf{H}_j < 2^{-21}(K_j)^{-2}$ for all $j \geq 0$.
- (2) $2^{75} \left(\frac{1}{\mathbf{H}_j}\right)^2 (K_j)^5 \exp\left(-\frac{1}{\mathbf{H}_j} 2^{-21}(K_j)^{-2}\right) \max\{1, \mathcal{Z}_j\}^2 < 1$ for all $j \geq 0$.
- (3) The sequence $(\delta_j)_{j \geq 0}$ given by

$$\delta_j = \sum_{l=j+1}^{\infty} \left\{ \prod_{k=j+1}^{l-1} 2^{28}(K_k)^3 \max\{1, \mathcal{Z}_k\} \right\} 2^{75} \left(\frac{1}{\mathbf{H}_l}\right)^2 (K_l)^5 \exp\left(-\frac{1}{\mathbf{H}_l} 2^{-21}(K_l)^{-2}\right) \max\{1, \mathcal{Z}_l\}^2$$

- satisfies $\delta_j \leq 2^{-3} \min\{2^{-1}\mathbf{H}_j, \mathcal{Z}_j\}$.
- (4) $\pi_0 \in S_3$ and $(\pi_j)_{j \geq 0}$ is the unique sequence in S_3 that satisfies $\pi_{j-1} = \Pi_L(\pi_j, \mathbf{f}_j)$ for all $j \geq 1$
- (5) $\sigma_* \in \{-1, +1\}^3$

Then $(\delta_j, \mathbf{f}_j) \in B_2\mathcal{F}$ for all $j \geq 0$ and there exists a sequence $(\mathbf{g}_j)_{j \geq 0}$ with $\mathbf{g}_j \in B[\delta_j, \mathbf{f}_j] \subset \mathcal{F}$ such that for all $j \geq 1$

$$\mathbf{g}_{j-1} = \Pi[\pi_j, \sigma_*](\mathbf{g}_j) \quad \text{and} \quad \pi_{j-1} = \mathcal{P}_L(\pi_j, \mathbf{g}_j)$$

Proof: Recall Proposition 4.4 and hypotheses (1) and (2) of the Theorem 6.2. We have

$$\begin{aligned}
2^{-2}\mathbf{H}_j &\leq \mathbf{H}_j \leq 2^2\mathbf{h}_j \\
2^{-1}\mathcal{Z}_j &\leq z_j \leq 2\mathcal{Z}_j \\
\max\left\{\frac{1}{w_j}, w_j, \frac{1}{q_j}, \frac{1}{|q_j-1|}, q_j\right\} &\leq 2^4 K_j \\
2^{-4}(K_j)^{-1} &\leq \min\{w_j, q_j, |q_j-1|\} \\
2\delta_j &\leq 2^{-1} \min\{w_j, q_j, |q_j-1|, \mathbf{h}_j, z_j\}
\end{aligned}$$

Hence, $B[2\delta_j, \mathbf{f}_j] \subset (0, \infty)^4$ for every $j \geq 0$. Further, $\forall(\mathbf{h}', w', q', z') \in B[2\delta_j, \mathbf{f}_j] \subset$

$(0, \infty)^4$, we have $q' \neq 1$ and

$$2^{-3}\mathbf{H}_j \leq 2^{-1}\mathbf{h}_j \leq \mathbf{h}_j - 2\delta_j \leq \mathbf{h}' \leq \mathbf{h} + 2\delta_j \leq 2\mathbf{h}_j \leq 2^3\mathbf{H}_j$$

and

$$2^{-2}\mathcal{Z}_j \leq 2^{-1}z_j \leq z_j - 2\delta_j \leq z' \leq z + 2\delta_j \leq 2z_j \leq 2^3\mathcal{Z}_j$$

and

$$\begin{aligned} \max\left\{\frac{1}{w'}, w', \frac{1}{q'}, \frac{1}{|q' - 1|}, q'\right\} &\leq \max\left\{\frac{1}{w' - 2\delta_j}, w' + 2\delta_j, \frac{1}{q' - 2\delta_j}, \frac{1}{|q' - 1| - 2\delta_j}, q' + 2\delta_j\right\} \\ &\leq 2\max\left\{\frac{1}{w_j}, w_j, \frac{1}{q_j}, \frac{1}{|q_j - 1|}, q_j\right\} \\ &\leq 2^5 K_j \end{aligned}$$

Recall the definition of τ_* from Section 3. Then the estimates above imply $\tau_*(\mathbf{h}', w', q', z') \geq 2^{-11}(K_j)^{-2}$, and by assumption (2) of the Theorem 6.2 we have

$$\mathbf{K}(\mathbf{h}', w', q', z') \leq 2^{75} \left(\frac{1}{\mathbf{H}_j}\right)^2 (K_j)^5 \exp\left(-\frac{1}{\mathbf{H}_j} 2^{-21} (K_j)^{-2}\right) \max\{1, \mathcal{Z}_j\}^2 < 1$$

Further, using the hypothesis (1) of the Theorem 6.2, we estimate

$$\mathbf{h}' \leq 2^3\mathbf{H}_j \leq 2^{-18}(K_j)^{-2} \leq 2^{-7}\tau_*(\mathbf{h}', w', q', z')$$

which is true for all $(\mathbf{h}', w', q', z') \in B[2\delta_j, \mathbf{f}_j]$. Therefore, $B[2\delta_j, \mathbf{f}_j] \subset \mathcal{F}$, $\forall j \geq 0$, in particular $\mathbf{f}_j \in \mathcal{F}$ (recall NewDef.3.19). We have $(\delta_j, \mathbf{f}_j) \in B_2\mathcal{F}$.

By the Proposition 4.4, we have $\mathbf{f}_{j-1} = \mathcal{Q}_L(\mathbf{f}_j) \forall j \geq 1$. Since $\delta_j \leq \delta_j$, Theorem 6.2 follows from Theorem 6.1. \square

Theorem 6.3. *Fix constants $\mathbf{D} \geq 1$, $\gamma \geq 0$. Suppose the vector $\mathbf{f}_0 = (\mathbf{h}_0, w_0, q_0, z_0) \in (0, \infty)^4$ satisfies*

- (1) $w_0 \in (0, 1) \setminus \mathbb{Q}$ and $q_0 \in (0, \infty) \setminus \mathbb{Q}$.
- (2) $k_n \leq \mathbf{D} \max\{1, n\}^\gamma$ with $(k_n)_{n \in \mathbb{Z}}$ as in Proposition 4.4, i.e.

$$(1 + q_0)^{-1} = \langle k_0, k_{-1}, k_{-2}, \dots \rangle \quad w_0 = \langle k_1, k_2, k_3, \dots \rangle$$

- (3) $0 < \mathbf{h}_0 < \mathbf{A}^\#$ where $\mathbf{A}^\# = \mathbf{A}^\#(\mathbf{D}, \gamma) = 2^{-56} \mathbf{D}^{-4} (4(\gamma + 1))^{-4(\gamma + 1)}$.

$$(4) \quad 0 \leq z_0 < \mathbf{A}^b \text{ where } \mathbf{A}^b = 2^{1-2\gamma} \frac{\mathbf{A}^{\sharp 2}}{\mathbf{h}_0}.$$

Then

- the assumptions of the Theorem 6.2 hold.
- Set $\rho_+ = \frac{1}{2}(1 + \sqrt{5})$. The sequence $(\delta_j)_{j \geq 0}$ in Theorem 6.2 satisfies for all $j \geq 0$:

$$(86) \quad \delta_j \leq \exp\left(-\frac{1}{\mathbf{h}_0} \mathbf{A}^{\sharp} \rho_+^{N(j)}\right) \quad \text{and} \quad N(j) \geq (\mathbf{D}^{-1}j)^{1/(\gamma+1)}$$

where $N : \mathbb{Z} \rightarrow \mathbb{Z}$ (Era Counter) is the map in Proposition 4.4

If $\gamma > 1$ and $\mathbf{D} > \frac{1}{\log 2} \frac{\gamma}{\gamma-1}$, then the set of all vectors $\mathbf{f}_0 \in (0, \infty)^4$ that satisfy (1), (2), (3), (4) has positive Lebesgue measure.

Proof.

Preliminaries. Fix $\mathbf{D} \geq 1$ and $\gamma \geq 0$ as in Theorem 6.3. For all $s = (s_1, s_2, s_3, s_4, s_5) \geq (0, 0, 0, 1, 0)$ with $s_i \in \mathbb{R}$, set

$$(87) \quad \mathbf{A}(s) = 2^{-s_1-s_2\gamma} \mathbf{D}^{-s_3} (s_4(\gamma+1))^{-s_5(\gamma+1)}$$

Observe that $\mathbf{A}(s)$ has properties $0 < \mathbf{A}(s) \leq 2^{-s_1} \leq 1$ and $\mathbf{A}(s) \leq \mathbf{A}(s')$ if $s \geq s'$.

Basic smallness assumptions. $k_n \leq \mathbf{D} \max\{1, n\}^\gamma$ for all $n \geq -2$ and $\mathbf{h}_0 < \mathbf{A}(\kappa)$. The vector $\kappa = (\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5) \geq (0, 0, 0, 1, 0)$ will be fixed during the proof.

Estimates 1. Recall Proposition 4.4. All the old estimates still hold (assuming $k_n \leq \mathbf{D} \max\{1, n\}^\gamma$ holds), namely

$$\begin{aligned} \mathbf{C}(w_0, q_0) &\leq 2\mathbf{D}^2(\gamma+1)^{2(\gamma+1)} = \mathbf{A}(1, 0, 2, 1, 2)^{-1} \\ J(n) &= \sum_{l=1}^n k_l \leq \mathbf{D} \sum_{l=1}^n l^\gamma \leq \mathbf{D} n^{\gamma+1} \\ j &\leq J(N(j)) \leq \mathbf{D} N(j)^{\gamma+1} \\ N(j) &\geq (\mathbf{D}^{-1}j)^{1/(\gamma+1)} \\ \mathbf{h}_j &\leq 2^4 \mathbf{h}_0 \rho_+^{-2N(j)} \\ \mathbf{h}_j &\geq 2^{-1} \mathbf{h}_0 \max\{1, 2\mathbf{D} N(j)^\gamma\}^{-2N(j)} \\ K_j &\leq \mathbf{D} 2^\gamma \max\{1, N(j)\}^\gamma \\ \mathbf{h}_j &\leq 2^{4+2\gamma} \mathbf{D}^2 \mathbf{h}_0 (\gamma+1)^{2(\gamma+1)} = \mathbf{h}_0 \mathbf{A}(4, 2, 2, 1, 2)^{-1} \end{aligned}$$

and additionally, we estimate

$$\begin{aligned}
\mathcal{Z}_j &= z_0 \prod_{l=1}^{N(j)-1} \frac{1}{w_{J(l)} w_{J(l-1)}} \\
&\leq z_0 \prod_{l=1}^{N(j)-1} (k_l + 1)(k_{l+1} + 1) \\
&\leq z_0 \prod_{l=1}^{N(j)-1} (2\mathbf{D}(l+1)^\gamma)^2 \\
&\leq z_0 \frac{1}{\max\{1, 2\mathbf{D}N(j)^\gamma\}^{-2N(j)}}
\end{aligned}$$

Require $\kappa \geq (25, 2, 2, 1, 2)$, then $\mathbf{h}_j < 2^{-21}(K_j)^{-2}$ and $\mathbf{h}_0 \leq 2^{-14}(\mathbf{C}_0(w_0, q_0))^{-1}$ as required by Theorem 6.2.

Estimates 2. Let $(\delta_j)_{j \geq 0}$ be as in Theorem 6.2. Then, with proper choice of κ , we have:

$$\begin{aligned}
\text{(A)} \quad \delta_{J(n)} &\leq 2^{-5} \frac{1}{\mathbf{h}_0} (2\mathbf{D}(n+1)^\gamma)^{-2(n+1)} \exp\left(-\frac{1}{\mathbf{h}_0} \mathbf{A}(\kappa) \rho_+^{n+1}\right) \\
\text{(B)} \quad \delta_j &\leq 2^{-3} \max\{2^{-1} \mathbf{H}_j, \mathcal{Z}_j\}, \text{ and } \delta_j \leq \exp\left(-\frac{1}{\mathbf{h}_0} \mathbf{A}(\kappa) \rho_+^{N(j)+1}\right)
\end{aligned}$$

The fact that (A) \Rightarrow (B) follows from the vacuum case is the direct consequence of the assumption (4) in Theorem 6.3 and definitions of \mathbf{H}_j and \mathcal{Z}_j from Theorem 6.2. Recall the following properties from Proposition 4.4: $N \circ J$ is the identity, and $J(m+1) = J(m) + k_{m+1}$.

Estimate for $n \geq 0$:

$$\begin{aligned}
\delta_{J(n)} &= \sum_{m=n}^{\infty} \sum_{l=J(m)+1}^{J(m+1)} \left\{ \prod_{k=J(n)+1}^{l-1} 2^{28} (K_k)^3 \max\{1, \mathcal{Z}_k\} \right\} 2^{75} \left(\frac{1}{\mathbf{h}_l} \right)^2 (K_l)^5 \exp \left(-(2^{21} \mathbf{h}_l K_l^2)^{-1} \right) \max\{1, \mathcal{Z}_l\}^2 \\
&\leq \sum_{m=n}^{\infty} \sum_{l=J(m)+1}^{J(m+1)} (2^{15} \max_{1 \leq k \leq l} K_k)^{3l+2} \left(\frac{1}{2\mathbf{h}_l} \right)^2 \exp \left(-(2^{21} \mathbf{h}_l K_l^2)^{-1} \right) \max_{1 \leq k \leq l} \{1, \mathcal{Z}_k\}^{l+2} \\
&\leq \sum_{m=n}^{\infty} k_{m+1} (2^{15+\gamma} \mathbf{D}(m+1)^\gamma)^{3J(m+1)+2} \frac{1}{2^2} \left(\frac{2}{\mathbf{h}_0 \max\{1, 2\mathbf{D}(m+1)^\gamma\}^{-2(m+1)}} \right) \\
&\times \exp \left(-2^{-25-2\gamma} \mathbf{D}^{-2} \frac{1}{\mathbf{h}_0} \rho_+^{2(m+1)} (m+1)^{-2\gamma} \right) \max\{1, z_0 (2\mathbf{D}(m+1)^\gamma)^{2(m+1)}\}^{J(m+1)+2} \\
&\leq \left(\frac{1}{\mathbf{h}_0} \right)^2 \sum_{m=n+1}^{\infty} (2^{15+\gamma} \mathbf{D} m^\gamma)^{10\mathbf{D} m^{\gamma+1}} \exp \left(-2^{-25-2\gamma} \mathbf{D}^{-2} \frac{1}{\mathbf{h}_0} \rho_+^{2m} m^{-2\gamma} \right) \max\{1, z_0 (2\mathbf{D} m^\gamma)^{2m}\}^{\mathbf{D} m^{\gamma+1}+2} \\
&\leq \left(\frac{1}{\mathbf{h}_0} \right)^2 \sum_{m=n+1}^{\infty} (2^{15+\gamma} \mathbf{D} m^\gamma)^{14\mathbf{D} m^{\gamma+2}} \exp \left(-2^{-25-2\gamma} \mathbf{D}^{-2} \frac{1}{\mathbf{h}_0} \rho_+^{2m} m^{-2\gamma} \right) \max\{1, z_0\}^{\mathbf{D} m^{\gamma+1}+2} \\
&\leq \left(\frac{1}{\mathbf{h}_0} \right)^2 \sum_{m=n+1}^{\infty} (2^{15+\gamma} \mathbf{D} m^\gamma)^{14\mathbf{D} m^{\gamma+2}} \exp \left(-2^{-25-2\gamma} \mathbf{D}^{-2} \frac{1}{\mathbf{h}_0} \rho_+^{2m} m^{-2\gamma} \right) \max\{1, z_0\}^{3\mathbf{D} m^{\gamma+1}}
\end{aligned}$$

Observe that $2^5 \frac{1}{\mathbf{h}_0} (2\mathbf{D}(n+1)^\gamma)^{2(n+1)} \leq \frac{1}{\mathbf{h}_0} (2^6 \mathbf{D} m^\gamma)^{2m}$ for all $m \geq n+1$. Therefore,

$$\begin{aligned}
\mathbf{S}(n) &\stackrel{\text{def}}{=} \delta_{J(n)} 2^5 \frac{1}{\mathbf{h}_0} (2\mathbf{D}(n+1)^\gamma)^{2(n+1)} \\
&\leq \left(\frac{1}{\mathbf{h}_0} \right)^3 \sum_{m=n+1}^{\infty} (2^{15+\gamma} \mathbf{D} m^\gamma)^{16\mathbf{D} m^{\gamma+2}} \exp \left(-2^{-25-2\gamma} \mathbf{D}^{-2} \frac{1}{\mathbf{h}_0} \rho_+^{2m} m^{-2\gamma} \right) \max\{1, z_0\}^{3\mathbf{D} m^{\gamma+1}} \\
&\leq \left(\frac{1}{\mathbf{h}_0} \right)^3 \sum_{m=n+1}^{\infty} \exp \left(16\mathbf{D} m^{\gamma+2} \log(2^{15+\gamma} \mathbf{D} m^\gamma) - 2^{-25-2\gamma} \mathbf{D}^{-2} \frac{1}{\mathbf{h}_0} \rho_+^{2m} m^{-2\gamma} \right) \max\{1, z_0\}^{3\mathbf{D} m^{\gamma+1}} \\
&\leq \left(\frac{1}{\mathbf{h}_0} \right)^3 \sum_{m=n+1}^{\infty} \exp \left(2^9 \mathbf{D}^2 (\gamma+1) m^{\gamma+3} - 2^{-25-2\gamma} \mathbf{D}^{-2} \frac{1}{\mathbf{h}_0} \rho_+^{2m} m^{-2\gamma} \right) \max\{1, z_0\}^{3\mathbf{D} m^{\gamma+1}} \\
&\leq \left(\frac{1}{\mathbf{h}_0} \right)^3 \sum_{m=n+1}^{\infty} \exp \left(2^9 \mathbf{D}^2 (\gamma+1) m^{\gamma+3} - 2^{-25-2\gamma} \mathbf{D}^{-2} \frac{1}{\mathbf{h}_0} \rho_+^{2m} m^{-2\gamma} + 3\mathbf{D} m^{\gamma+1} \log(\max\{1, z_0\}) \right)
\end{aligned}$$

Observe that if we require $\kappa \geq (35, 2, 4, \frac{3}{2}, 3)$, the absolute value of the second term is at least twice the absolute value of the first term, that is,

$$\begin{aligned}
2 \cdot 2^9 \mathbf{D}^2(\gamma+1)m^{\gamma+3} \cdot 2^{25+2\gamma} \mathbf{D}^2 \rho_+^{-2m} m^{2\gamma} &= 2^{35+2\gamma} \mathbf{D}^4(\gamma+1) \rho_+^{-2m} m^{3\gamma+3} \\
&\leq 2^{35+2\gamma} \mathbf{D}^4 \left(\frac{3}{2}(\gamma+1) \right)^{3(\gamma+1)} \\
&= \mathbf{A}(35, 2, 4, \frac{3}{2}, 3)^{-1} \\
&\leq \mathbf{A}(\kappa)^{-1} \\
&\leq \frac{1}{\mathbf{h}_0}
\end{aligned}$$

Therefore,

$$\mathbf{S}(n) \leq \left(\frac{1}{\mathbf{h}_0} \right)^3 \sum_{m=n+1}^{\infty} \exp \left(-2^{-26-2\gamma} \mathbf{D}^{-2} \frac{1}{\mathbf{h}_0} \rho_+^{2m} m^{-2\gamma} + 3\mathbf{D} m^{\gamma+1} z_0 \right)$$

Now, using the estimate $2^{26+2\gamma} \mathbf{D}^2 \sup_{m \geq 1} \rho_+^{-m} m^{2\gamma} \leq 2^{26+2\gamma} \mathbf{D}^2 (2(\gamma+1))^{2(\gamma+1)} = 2^{-2} \mathbf{A}_*^{-1}$ with $\mathbf{A}_* = \mathbf{A}(28, 2, 2, 2, 2)$, and requiring $\kappa \geq (28, 2, 2, 2, 2)$, we get $\mathbf{h}_0 \leq \mathbf{A}_*$.

Further,

$$3\mathbf{D} m^{\gamma+1} z_0 = 3\mathbf{D} \rho_+^m \rho_+^{-m} m^{\gamma+1} z_0 \leq 3\mathbf{D} \rho_+^m (\gamma+1)^{\gamma+1} z_0$$

and

$$3\mathbf{D}(\gamma+1)^{\gamma+1} < 2^2 \mathbf{D}(\gamma+1)^{\gamma+1} = \mathbf{A}(2, 0, 1, 1, 1)^{-1} \leq \mathbf{A}_*^{-1}$$

Therefore,

$$\begin{aligned}
\mathbf{S}(n) &\leq \left(\frac{1}{\mathbf{h}_0} \right)^3 \sum_{m=n+1}^{\infty} \exp \left(-4 \frac{1}{\mathbf{h}_0} \mathbf{A}_* \rho_+^m + z_0 \frac{1}{\mathbf{A}_*} \rho_+^m \right) \\
&\leq \left(\frac{1}{\mathbf{h}_0} \right)^3 \sum_{m=n+1}^{\infty} \exp \left(-4 \rho_+^m \left(\frac{1}{\mathbf{h}_0} \mathbf{A}_* - z_0 \frac{1}{4\mathbf{A}_*} \right) \right)
\end{aligned}$$

By assumption (4) in Theorem 6.3, we have $z_0 < 2\mathbf{A}_*^2/\mathbf{h}_0$. Therefore,

$$\mathbf{S}(n) \leq \left(\frac{1}{\mathbf{h}_0} \right)^3 \exp \left(-\frac{1}{\mathbf{h}_0} \mathbf{A}_* \rho_+^{n+1} \right) \exp \left(-\frac{1}{\mathbf{h}_0} \mathbf{A}_* \right) \sum_{m=1}^{\infty} \exp(-2\rho_+^m)$$

Using $\sum_{m=1}^{\infty} \exp(-2\rho_+^m) \leq 1$ and requiring $\kappa \geq (56, 4, 4, 2, 4)$, we get $\mathbf{h}_0 \leq \mathbf{A}_*^2$ and

$\left(\frac{1}{\mathbf{h}_0}\right)^3 \exp\left(-\frac{1}{\mathbf{h}_0} \mathbf{A}_*\right) \leq 1$. Therefore,

$$\mathbf{S}(n) \leq \exp\left(-\frac{1}{\mathbf{h}_0} \mathbf{A}_* \rho_+^{n+1}\right)$$

The rest of the argument is identical with the vacuum case.

Lebesgue measure of the set of admissible \mathbf{f}_0 . The set of all $(\mathbf{h}_0, w_0, q_0) \in (0, \infty)^3$ that satisfy the assumptions (1)-(4) of the Theorem 6.3, is a product $(0, \mathbf{A}^\sharp) \times F_w \times F_q$ with $F_w \subset (0, 1) \setminus \mathbb{Q}$, $F_q \subset (0, \infty) \setminus \mathbb{Q}$. Further, for z_0 , F_z is the area under \mathbf{A}^b . Note that $(0, \mathbf{A}^\sharp)$, F_q and F_z have positive measure.

The rest of the proof stays the same as for the vacuum case in [2]. Suppose $\gamma > 1$ and $\mathbf{D} > \frac{1}{\log 2} \frac{\gamma}{\gamma-1}$. Let $G(x) = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$ be the Gauss map from $(0, 1) \setminus \mathbb{Q}$ to itself. Let μ_G be the probability measure on $(0, 1) \setminus \mathbb{Q}$ with $d\mu_G(x) = \frac{1}{(1+x)\log 2} dx$, and with a well-known property $\mu_G(X) = \mu_G(G^{-1}(X))$ for all measurable $X \subset (0, 1 \setminus \mathbb{Q})$. Observe that $k_{n+1} = \lfloor \frac{1}{G^n(w_0)} \rfloor$ for all $n \geq 0$.

For all $n \geq 0$ define

$$\mu_G(X_n) = \mu_G((0, \mathbf{D}^{-1}(n+1)^{-\gamma}) \setminus \mathbb{Q}) = \frac{1}{\log 2} \log \left(1 + \frac{1}{\mathbf{D}(n+1)^\gamma}\right) \leq \frac{1}{\log 2} \frac{1}{\mathbf{D}(n+1)^\gamma}$$

and let X_n^c be the complement of X_n in $(0, 1) \setminus \mathbb{Q}$. We then have $\bigcap_{n \geq 0} X_n^c \subset F_w$ and

$$\mu_G(F_w) \geq \mu_G\left(\bigcap_{n \geq 0} X_n^c\right) = 1 - \mu_G\left(\bigcup_{n \geq 0} X_n\right) \geq 1 - \sum_{n \geq 0} \mu_G(X_n) \geq 1 - \frac{1}{\mathbf{D} \log 2} \frac{\gamma}{\gamma-1} > 0$$

Positive Gauss measure implies positive Lebesgue measure.

□.

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